Optimal shapes in boundary value problems

OptShaBVP

Work Program

Principal Investigator: **Davide Zucco**

Host Institution: **Politecnico di Torino - Dipartimento di Scienze Matematiche**

Proposal Title: **Optimal shapes in boundary value problems**

Proposal Acronym: **OptShaBVP**

Proposal Duration: **12 month**

ERC domain: **PE1_8–Analysis**

Abstract

The project will focus on selected problems related to shape optimization problems for boundary value problems, namely: optimal shapes for the eigenvalues of elliptic operators, optimal nucleation of dislocations, quantitative estimates for irrigation functionals.

The analysis of these points leads to deep mathematical questions originated by a common feature: these quantities strongly depend on the shape over which they are defined. Therefore, one is interested in the "best" domain to optimize these quantities.

Most of these functionals have a *variational* structure. Therefore, their analysis need advanced mathematical tools from the *calculus of variations*, from *measure theory* and *geometric measure theory*, from *spectral theory*, from the theory of *nonlinear elliptic partial differential equations*, and also from *metric geometry*. Our goal is to develop new mathematical tools in these areas for the study of selected problems related to boundary value problems.

1 Introduction

Variational problems for the eigenvalues of the Laplace operator have a long history, which dates back to 1877 when Lord Rayleigh observed and conjectured that of all membranes with a given area, the circle has the minimum principal frequency. Indeed, prior to this date B. de Saint-Venant had conjectured (1856) that of all cross-sections with a given area, the circle has the maximum torsional rigidity, but strictly speaking this variational problem cannot be formulated in terms of eigenvalues of the Laplace operator.

Lord Rayleigh's conjecture was proved many years later by G. Faber in 1923 and, independently, by E. Krahn in 1924, and many other contributions to similar problems were given in the early 20's by several distinguished mathematicians such as J. Hadamard, H. Poincaré, T. Carleman and G. Szegő just to mention a few.

The so called *isoperimetric inequalities* for eigenvalues (and, more generally, for other geometric or physical quantities which are naturally defined in a variational form) have become an important research area in mathematics, especially after the great impulse given by the now classical book of Pólya and Szegő [23].

In the last decades, a new branch of research has been developed named *shape optimization*, which essentially embraces all those minimization problems where the unknown is a domain (the shape) in Euclidean space, subject to some constraints (see for example [4, 13, 14, 15, 24]). And, of course, variational problems for boundary value problems fit well into this framework.

As *boundary value problem* we refer to a differential equation defined on a domain of the euclidean space, with supplementary *boundary conditions*, which are usually prescribed on a specific region of the domain (e.g., the whole boundary of the domain). The solutions of the boundary value problem solve the differential equation and satisfy the boundary conditions (these are formalized within suitable functional spaces). One of the most studied boundary conditions are the so-called *Dirichlet boundary conditions*: one requires to solutions to assume specific values over a suitable region of the domain. Since the solutions depend on the region where the boundary conditions are imposed, from these one can construct several shape functionals, such as the *compliance* or an *eigenvalue* of a suitable operator.

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Most of these functionals have a *variational* structure. Therefore, their analysis need advanced mathematical tools from the *calculus of variations*, from *measure theory* and *geometric measure theory*, from *spectral theory*, from the theory of *nonlinear elliptic partial differential equations*, and also from *metric geometry*. Our goal is to develop new mathematical tools in these areas for the study of the selected problems.

2 State of the art and objectives

The starting point of this research project is an optimization problem introduced in my PhD thesis. In [31] we studied the problem of finding, in a domain of the plane, the "best" – in shape and location – region that maximizes the first eigenvalue of an elliptic operator in divergence form (possibly with nonconstant coefficients), with Dirichlet conditions imposed along the region (similar problems have been considered in [11, 12], but for a region of *fixed* shape, so that only its placement is to be optimized). The Dirichlet region is a compact and connected set with prescribed length, in the sense of one-dimensional Hausdorff measure. Of course, when the length constraint increases, the optimal configurations tend to saturate the domain, and in the limit any information concerning the distribution of these configurations is lost. We investigated in [28] e [29], with P. Tilli, how this saturation occurs, answering questions such as: with what limit density (length per unit area) the maximizers saturate the domain? With what local orientation? Indeed, *via* Γ-convergence of suitable functionals defined over the space of probability measures (and, more generally, on the space of *varifolds*) it is possible to keep track of the density of the optimal configurations, as well as their local orientation.

This problem has an interesting physical interpretation. The domain represents an elastic structure (e.g., a membrane) in the plane which is fixed along its boundary and, as such, has a fundamental frequency given by the square root of the first eigenvalue of a suitable operator. One is interested in augmenting and possibly maximizing this fundamental frequency, by fixing the membrane along a supplementary curve (or system of curves) of given total length, which may be placed anywhere in the domain. One may think to the region as a sort of stiffening rib, to obtain a reinforced structure. Note that not only the location but also the shape of the region is free.

An extension of this problem to a larger class of admissible configurations has been considered in [16] by the PI with the collaboration of A. Henrot. We maximized the first eigenvalue over sets of finite *outer Minkowski content* (including curves and system of curves, but also sets with possibly positive measure). In this framework we were able to prove some of the qualitative properties satisfied by the maximizers, passing toward regularity and symmetry results. Moreover, for specific domains (i.e., disks, rings, and, more generally, disks with several holes) we identified the unique maximizer.

This problem leads to several directions of research. The principal objectives of this research project are the following.

1. **Qualitative properties of the maximizers.** We want to investigate the regularity of the maximizer for the problem studied in [29]. According to a result in [31] the optimal configurations are *Alfhors regular*. We intend to study additional regularity results for the optimal configurations, as well as topological and geometrical aspects. Moreover, we will study the asymptotic behavior of the optimal configurations as the length constraint tends to zero.

2. **Different spacial dimensions.** The problem so far (see [28, 29]) has been treated only up to dimension 2 , and extensions to higher dimensions should be possible. On the other hand, in dimension 1 (see [30]), several connections with other fields like *Sturm-Liouville* and *optimal partition* problems are possible (see [9, 17]).

3. **Functions of higher eigenvalues.** In the one dimensional case [30], we studied the minimization of the sum of inverse Dirichlet eigenvalues. Other generalizations could be considered, e.g. ratio or sums of eigenvalues, see [1], involving the higher Dirichlet eigenvalues.

4. **Schrödinger eigenvalues.** The eigenvalue problem for Schrödinger operators is also treated in [31]. More precisely we showed that the potential does not change the asymptotics of the maximizers. Moreover it could be interesting to study the effect of the potential on the maximizers, whenever the Dirichlet region is fixed.

5. **Applications: compliance, dislocations, bridges, irrigations.** We will extend this study to other functionals, more involved in the applications. We will study problem [5] in the case of an anisotropic compliance. Then qualitative properties of a model for dislocations introduced in [18] will be investigated. Moreover, a similar maximization problem could be studied for the eigenvalues of the clamped plate or the buckled plate equations (see Ashbaugh-Benguria [1] e Ashbaugh-Bucur [2]). For the stability of suspended bridges, we initiated a work in [3]. We aim at developing further results in this direction. Finally, in [27] there are explicit solutions to optimal shapes for an irrigation problem. We will study quantitative estimates for this problem.

6. **Other boundary conditions: Neumann, Robin, Steklov.** An analogous problem to [29] could be treated for different boundary conditions, such as Neumann, Robin and Stekloff. One of the tools involved in the Dirichlet case is the monotonicity property of the functional with respect to domain inclusion. Since in general the monotonicity formula is no longer valid, it might be necessary to develop new ideas. However, in some cases it is possible to control this drawback. Indeed, for Neumann boundary conditions in [25] it has been studied the asymptotic of the *Mumford-Shah functional* (see also [10] for a related problem on the *thick Neumann's sieve*). For this, the study will begin with the generalization of [25] to the anisotropic case.

7. **Geodesics in manifolds with boundary.** We will study the relation between different notions of convex envelope in a manifold with boundary (such as an euclidean

bounded domain). More precisely, we will introduce and investigate the notion of: *local convex set*, *global convex set*, *geodesic set* and *set with minimal perimeter*. This topic will be useful for further developments of the theory on boundary values problems.

3 Methodology

The correlated deliverables to the objectives of the work package described in section 2 are the following.

- **About 1.** To obtain regularity and geometrical results on the optimal configurations we will adapt some of the ideas, recently developed for the minimizer of the compliance in [8]. In particular, we will use a *blow-up* analysis, similar to the one developed in [26], for the irrigation problem.

- **About 2.** In higher spacial dimension the generalization seems to be even more delicate. There seem to be two natural ways of extending the problem to higher dimension:

  A) replace 1-dimensional sets for the Dirichlet region with $(n-1)$-surfaces of given area;

  B) still work with 1-dimensional Dirichlet regions in dimension n, but work with the first eigenvalue of the p-laplacian in the space $W^{1,p}$, with $p > n-1$ so that a Dirichlet condition along a 1-dimensional set is meaningful.

  Extension B) is certainly possible (and probably the extension would be a purely technical one, see [21, 22]). On the other hand, extension A) seems hard to obtain, since semicontinuity and compactness results (Blaschke & Gołab theorems for 1-dimensional Hausdorff measure) fail for surfaces (i.e. for n-dimensional Hausdorff measure with n¿1). A possible variant to A) is of working with a finite union of n-balls, as treated for a related problem in [6].

- **About 3.** One way for reflecting the effect of the potential in the Γ-limit, might be to rescale it appropriately.

- **About 3., 4. and 5.** To deal with anisotropic functionals we will use the class of varifolds. This extension will be not a merely technical one, since we will need to combine different techniques developed in [29], [5] and in [25]. In particular, we expect as Γ-limit an anisotropic local functional (i.e., the integral over the domain of a quadratic form dictated by the coefficients of the elliptic equation).

- **About 5.** The asymptotics of the dislocations as the number of points tends to infinity will be developed as in [5]. We expect a Γ-limit that is minimize by a measure that it is not concentrated on the boundary of the domain. Moreover, for constant forces acting on circular materials, it will be proved that it is energetically convenient for one dislocation to lie in the center of the material.

- **About 6.** By means of classical results from *global analysis* and suitable *asymmetry* functionals (see, e.g., [19]), we will improve some estimates proved in [27]. In particular, we will analyze a quantitative estimate for the solution of the considered irrigation problem, refining a lower bound for the area of the tubular neighborhood of a continuum in the euclidean space.

- **About 7.** In the simply connected case, a theorem *à la* Cartan-Hadamard, adapted to manifolds with boundary, will allow to establish the equivalence between the notions of gedesical and local convexity. However, in the non-simply connected case we will prove the following chain of implication. Geodesical convexity imply local convexity which imply minimality of the perimeter.

4 Resources

The financial support requested will foster the collaboration among the team members on the subjects of the project and will promote new scientific contacts with other groups working on similar problems with different expertise. These objectives will also be obtained through the interactions with the researchers of the Department of Mathematical Sciences of the Polytechnic University of Turin and with the participation to international conferences. Moreover, this research project will allow to consolidate some collaborations with researchers of the Department of Mathematics, University of Turin, strengthening connection between these two mathematical institutions of the city.

4.1 Principal Investigator and team members

The *Principal Investigator* (PI) – Davide Zucco – will coordinate the entire project and will have full responsibility of the project development.

The scientific project will also be developed with the contribution of the following *team members*: Riccardo Adami, Enrico Serra, Paolo Tilli (Politecnico di Torino), Elena Cordero (Università di Torino), Gianni Dal Maso, Guido De Philippis (SISSA, Trieste), Dario Mazzoleni (Università di Pavia), Giovanni Franzina (Università di Roma "La Sapienza"), Antoine Henrot, Ilaria Lucardesi (Université de Lorraine), Riccardo Scala (Universidade de Lisboa).

4.2 Budget

The total funds 35.000,00€ will be distributed to cover the following costs.

- **Post-doc**. The largest budget item is for a post-doctoral position for the PI. The cost for a post-doc fellowship is 24.000,00€.
- **Travel**. The travel and subsistence expenses for the PI and the team members will be for participation in conferences and for scientific collaborations on subjects related to the project. The average estimated cost is 3.000,00€.
- **Workshop.** We plan to organize one international conference and to spend 5.000,00€ for the invitation of the main speakers.
- **Visitors.** We plan to invite some external experts at Polytechnic University of Turin. These invitations will strengthen the scientific links with other research groups in these fields or in related areas and will also allow us to obtain updated informations on the most recent advances on topics related to the project. The travel expenses for short term invitations of external experts are 3.000,00€.

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