PDES, FREE BOUNDARIES, NONLOCAL EQUATIONS AND APPLICATIONS

RESEARCH PROJECT BY SERENA DIPIERRO

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1. Framework

The proposed research project focuses on elliptic partial differential equations, also from the perspective of free boundary problems and nonlocal equations of fractional type. The topic lies in the field of mathematical analysis, but it is truly interdisciplinary in nature, since many problems (such as the nonlocal minimal surfaces) require a deep geometric insight, others (such as the phase transitions and the theory of atom dislocations) are motivated by questions in mathematical physics and material sciences, others (such as the long-range Ising models) are related to statistical mechanics, others (such as the thin obstacle problem) are related to elasticity theory and engineering, and others (such as the nonlocal logistic map or the transmission models in neurons) have important applications in biology and population dynamics.

The perspective that we take in this project is very broad, since the nonlocal behavior that we take into account can arise from fractional operators, or in view of L^∞-oscillation terms in the energy functional. The first approach is very related to singular integrals, interpolation spaces and anomalous diffusion, the second approach has deep links with convex analysis and geometric measure theory.

In many concrete cases, the nonlocal features of the problems are produced by memory effects, long-range particle interactions, boundary effects, or ramifications and porosity of the transmission media, thus leading to fractional diffusion in either time or space.

Investigating these topics is therefore of great importance both for the research in pure mathematics and in view of a better understanding of many real world phenomena, which also offer a great variety of potential applications.

2. Methodology

Due to the complexity and the transversal connections of the problems under consideration, the method of investigation will rely on several advanced mathematical theories, such as elliptic regularity, blow-up and blow-down methods and functional analysis.

Methods from the calculus of variations will be used to discuss the main properties of the energy functionals, techniques of nonlinear analysis will be applied for establishing the existence (and possible multiplicity) of the solutions, a creative use of geometric analysis will be crucial to understand some of the new objects that have recently appeared in the mathematical literature (such as nonlocal geodesics, nonlocal minimal surfaces and nonlocal mean curvatures), and tools of geometric measure theory will play a crucial role in the development of the regularity theory.

Of course, a strong sensibility for the applications and a vivid physical intuition will be essential to efficiently promote the connections with real-world phenomena. For this, we also aim to intensify the interactions with scientists of different fields, so to favor also the formation of unified visions and a common scientific language to deal with the complex problems of contemporary research. For this specific goal, we also aim to organize an international conference with a multi-disciplinary flavor.

3. Mathematical problems

We provide here a short list of some of the mathematical problems that we would like to investigate. Of course, we expect that this list will be widely enlarged by the development of the project itself, which will open new lines of investigation and propose new challenging problems, thus enlarging the horizons of the research plan.

3.1. **Nonlocal Dirichlet forms.**

The classical Dirichlet energy

$$\mathcal{D}_1(u) := \int_\Omega |\nabla u(x)|^2 \, dx \tag{1}$$

takes into account the "local" oscillation of the functional and it models, for instance, the displacement of an elastic membrane fixed at the boundary.

A careful study of the minimizers of the Dirichlet energy is essential in the analysis of free boundary problems, since the notion of "harmonic replacement" provides fundamental energy competitors (see e.g. [5]).

The functional in (1) is, from the point of view of functional analysis, the seminorm in the Sobolev space $H^1(\Omega)$. Hence, a natural object to take into account is a fractional version of $\mathcal{D}_1$, that corresponds to the seminorm in the Sobolev space $H^s(\Omega)$, for $s \in (0,1)$, with $\Omega \subset \mathbb{R}^n$, namely

$$\mathcal{D}_s(u) := \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy, \tag{2}$$

where $\Omega^c := \mathbb{R}^n \setminus \Omega$. The domain of integration in (2) is chosen to efficiently comprise the external data in the nonlocal case: roughly speaking, the domain $\Omega = \mathbb{R}^n \setminus \Omega^c$ in (1) corresponds to the complement of the set where the data are prescribed; since the data prescription in the nonlocal case are naturally prescribed for $(x, y) \in \Omega^c \times \Omega^c = (\Omega^c)^2$, the domain of integration in (2) can be seen as the complementary set of data prescription, in perfect "conceptual agreement" with the classical case.

An equivalent formulation of (2) can be written in terms of Muckenhoupt weights in the extended halfspace $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ as

$$\widetilde{\mathcal{D}}_s(u) := \int_{\Omega^+} z^{1-2s} \, |\nabla U(x,z)|^2 \, dx \, dz, \tag{3}$$

with $x \in \mathbb{R}^n$, $z > 0$ and $\Omega^+ \subset \mathbb{R}^{n+1}_+$.

The harmonic replacements induced by the energy functionals in (2) and (3) have been studied in details in [16–18] and lead to the foundation of the regularity theory of a series of fractional free boundary problems.

Our next goal is to consider Dirichlet energy "of L^∞-oscillation type". Namely, given $r > 0$, we consider the oscillation function

$$\operatorname{osc}(u, r, x) := \sup_{B_r(x)} u - \inf_{B_r(x)} u$$

and the corresponding nonlocal energy functional

$$\mathcal{D}^\star_r(u) := \int_\Omega \operatorname{osc}(u, r, x)\, dx. \tag{4}$$

Differently from (1) and (2), the functional in (4) deals with the global behaviour of the oscillation of a function and not with an "averaged oscillation". The geometric counterpart of this analytic feature is that the functional in (4) is not scale invariant, so new ideas will be needed to address, at small scale, regularity and monotonicity theories. In some sense, we expect that, at large scale (corresponding to small r), the oscillation will "mimic" the features of the gradient, thus endowing the problem with "good properties", but on finer scales (comparable to or smaller than r), the oscillation may not be affected by discontinuities and minimizers may exhibit very different structures.

In addition, the functional in (4) does not possess any associated extended problem of local type, therefore many classical techniques related to scaled iterations and monotonicity formulas are not easily applicable in this setting.

Making these observations rigorous and developing a consistent variational theory is one of the goals of this project. In particular, we want to study:

- local minimizers with respect to fixed data outside a domain Ω,
- local minimizers with respect to fixed data in a neighborhood of size r of Ω^c,
- class A minimizers (local minimizers in any ball: classification results, construction of class A minimizers, periodic and quasi-periodic structures),
- properties of the harmonic replacements with respect to the functional in (4),
- phase transition models related to the functional in (4),
- free boundary problems related to the functional in (4).

In particular, we want to obtain quantitative and qualitative results for minimizers and critical points of the energy functional, monotonicity of minimizers in terms of the data, multiplicity issues, periodicity of the minimizers and of the critical points, plane-like minimizers (i.e. minimizers which remain at a bounded distance from a linear function).

We also want to relate these types of problems with classical models in statistical mechanics, such as the long-range Ising model.

3.2. **Nonlocal perimeter functionals.** As detailed in [12, 13], the nonlocal Dirichlet form in (4) can be related, via a nonlocal co-area formula, to a nonlocal perimeter functional inspired by the so-called "Minkowski content", that is the volume of the r-neighborhood of the boundary of a set E, namely

$$\mathcal{L}^n\Big(\{x \in \Omega \text{ s.t. } B_r(x) \cap (\partial E) \neq \varnothing\}\Big). \tag{5}$$

For small r, this functional recovers the classical perimeter, while for large r it behaves like a classical volume term. In this interpolation between volume and perimeter, the functional in (5) resembles the nonlocal perimeter introduced in [6]. Nevertheless, the functional in (5) is not scaling invariant, differently from that in [6], hence the classical blow-up methods and the monotonicity formulas cannot be used in this framework. For this reason, one may expect the behaviors of minimizers and critical points of (5) to be highly influenced by the scale parameter of interaction r. Our investigation will focus on:

- global and relative isoperimetric inequalities,
- quantitative isoperimetric inequalities,
- capillarity problems,
- boundary behaviors,
- graph properties,
- density estimates,
- asymptotic properties for small and large r,
- classification results,

- monotonicity formulas with remainders,
- geometric motions.

3.3. **Nonlocal and nonlinear free boundary problems.** Inspired by the classical free boundary problems in [1, 2] and by the recent developments in nonlocal free boundary problems in [7, 8, 15], we would like to take into account the possibly nonlinear superposition of a Dirichlet energy and a bulk energy of volume type. Different types of problems can be written in the form

$$\Phi\big(\mathcal{D}(u)\big) + \Psi\big(\mathcal{P}(E)\big). \tag{6}$$

Here, Φ and Ψ denote suitable (possibly nonlinear) functions, $\mathcal{D}(u)$ is an energy of Dirichlet type and $\mathcal{P}(E)$ is a bulk energy.

Concretely, one can study the cases in which the Dirichlet energy is that in (1), (2) or (4), and the bulk energy could be of perimeter, fractional perimeter or volume type, or being as in (5). Of course, in spite of the general form of the functional in (6), all these cases will provide very different problems, due to the different scale invariance involved, or for the complete lack of scale invariance. Therefore, depending on the case into account, we would like to either establish new regularity results or provide counterexamples. The lack of scale invariance in the nonlinear case is also expected to produce severe instabilities of minimizers in dependence of the size of the domain.

We would also like to address cases in which the energy functional is not differentiable, thus providing a lack of maximum principle. In this setting, a model for the classical case is that studied in [3] and recent contributions in the fractional framework has been given in [28], in terms of Hölder continuity of the minimizers. Here, we want to obtain optimal regularity results for the free boundary, using extension techniques.

3.4. **Anomalous diffusion in space and time.** From the probabilistic point of view, anomalous diffusion in space, when related to fractional scaling, comes from stable stochastic processes, while anomalous diffusion in time is often motivated by non-Markovian processes with memory processes and related to the Riemann-Liouville or to the Caputo derivatives, see [11].

Related to these models, we would like to study equations of the type

$${}^{C}D^{\alpha}_{t} u(x,t) + (-\Delta)^s u(x,t) = f\big(x,t,u(x,t)\big).$$

Here, ${}^{C}D^{\alpha}_{t}$ represents the Caputo derivative of order $\alpha \in (0,1)$, namely

$${}^{C}D^{\alpha}_{t} v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot v(\tau)\,d\tau}{(t-\tau)^\alpha}$$

and $(-\Delta)^s$ is the classical Laplacian when $s = 1$ and the fractional Laplacian when $s \in (0,1)$ – that is, in Fourier space

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\hat u.$$

Here above, Γ denotes the Gamma-function.

First of all, we would like to establish general theories for ordinary differential equations of Caputo type, extending the results in [23, 24, 29]. In general, explicit solutions are hard to find in this case, and a deep knowledge of special functions is needed to deal with particular cases. Also, the memory effect complicates the use of comparison principles, which fail at small scales.

In addition, we would like to consider nonlocal equations, of mono-stable or bi-stable type, and logistic equations motivated from biology. In particular, inspired by the classical work of [4] and by the case of spatial anomalous diffusion developed in [9, 10], we would like to obtain asymptotic estimates for solutions of nonlinear equations of Fisher-KPP type, say when the nonlinearity is of the semilinaer type $f(u) = u(1-u)$, detecting the speed of invasion. This is an extremely challenging problem, for which, first, a careful analysis of the one-dimensional case is mandatory. We hope to overcome the difficulties provided by the memory effect by the introduction of novel gluing techniques, which can estimate asymptotically the error produced by the lack of translation invariance.

3.5. **Regularity and homogeneity for nonlocal systems.** We want to study systems of the type

$$\int_{\mathbb{R}^n} \frac{a(x,y)\,(u(x)-u(y))}{|x-y|^{n+2s}}\,dy = 0,$$

where the integral is taken in the principal value sense, $a(x,y)$ is a positive definite and bounded matrix, $s \in (0,1)$ and $u : \mathbb{R}^n \to \mathbb{R}^n$ (or, more generally, $u : \mathbb{R}^n \to \mathbb{R}^m$). We want to classify homogeneous solutions. In addition, we want to construct a matrix $a(x,y)$ for which the "normal" direction $\frac{x}{|x|}$ is a solution (for $n \geq 2$). This example will show that fractional elliptic systems exhibit discontinuous solutions.

The proof will follow these lines: first, we construct a degenerate matrix which solves the equation (taking advantage of the geometric properties of the solution); then, we perturb such matrix making use of a (fractional) divergence free vector field.

This project is the fractional counterpart of the regularity for elliptic systems of the form

$$\mathrm{div}\,(A\nabla u) = 0,$$

with $A = A(x)$ a matrix, see e.g. [20] and [21], and our counterexample aims at being the fractional analogue of a celebrated one by De Giorgi in [14], see also [22].

Other counterexamples to regularity theory should be seen as the counterpart of those in [26] and [27], also in light of the recent geometric approach in low dimension provided in [25].

The fractional case that we want to address will be very challenging, especially because "explicit" computations are almost impossible. Our strategy will be to exploit the geometry of the problem as much as possible and use expansions with respect to the fractional parameter. In this way, we hope to be able to construct counterexamples also in dimension lower than in the classical case. The reason for this dimension improvement lies in the fact that, in the classical examples, dimension 2 is usually excluded due to a degeneracy given by a constant of the type $n-2$. We believe that such degeneracy gets removed in the fractional case, by looking at the second order expansion in the fractional parameter (hence, in this case, our aim is to obtain ellipticity constants of the type $(n-2)+(1-s)c$, with $c > 0$, which would be strictly positive also in dimension 2).

3.6. **Free boundary problems for higher order operators.** We aim at studying free boundary problems obtained by the minimization of the energy functional

$$\int_{\Omega} |\Delta u(x)|^2\,dx + \mathcal{L}^n\big(\Omega \cap \{u > 0\}\big).$$

Such functional can be seen as a higher order analogue of that in [1]. In this case, the lack of the maximum principle (typical of the case of fourth order equations, see e.g. [19]) makes the classical regularity theory not available. We plan to start first with the case of Navier boundary conditions (i.e., prescribing u and Δu along $\partial\Omega$) and focus on the two-dimensional case, to obtain regularity results of the solution and of the free boundary. The free boundary condition in this case involves higher derivatives, but the regularity of the solution will lead to significant simplifications at least in the one-phase case (in which the Laplacian should reduce to being constant along the free boundary).

4. Additional activities

The project will hire a PostDoc with a competitive and internationally advertised opening.

We expect to obtain several high quality results, to be published in top international journals.

We also foresee a very intense dissemination activity, based on seminars and conferences, which will make the results of the project available to the international scientific community.

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. MR618549

[2] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461, DOI 10.2307/1999245. MR732100

[3] H. W. Alt and D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math. **368** (1986), 63–107. MR850615

[4] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math. **30** (1978), no. 1, 33–76, DOI 10.1016/0001-8708(78)90130-5. MR511740

[5] I. Athanasopoulos, L. A. Caffarelli, C. Kenig, and S. Salsa, *An area-Dirichlet integral minimization problem*, Comm. Pure Appl. Math. **54** (2001), no. 4, 479–499, DOI 10.1002/1097-0312(200104)54:4¡479::AID-CPA3¿3.3.CO;2-U. MR1808651

[6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144, DOI 10.1002/cpa.20331. MR2675483

[7] Luis A. Caffarelli, Jean-Michel Roquejoffre, and Yannick Sire, *Variational problems for free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1151–1179, DOI 10.4171/JEMS/226. MR2677613

[8] Luis Caffarelli, Ovidiu Savin, and Enrico Valdinoci, *Minimization of a fractional perimeter-Dirichlet integral functional*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 4, 901–924, DOI 10.1016/j.anihpc.2014.04.004. MR3390089

[9] Xavier Cabré, Anne-Charline Coulon, and Jean-Michel Roquejoffre, *Propagation in Fisher-KPP type equations with fractional diffusion in periodic media*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 19-20, 885–890, DOI 10.1016/j.crma.2012.10.007 (English, with English and French summaries). MR2990897

[10] Xavier Cabré and Jean-Michel Roquejoffre, *The influence of fractional diffusion in Fisher-KPP equations*, Comm. Math. Phys. **320** (2013), no. 3, 679–722, DOI 10.1007/s00220-013-1682-5. MR3057187

[11] Michele Caputo, *Linear model of dissipation whose Q is almost frequency independent. II*, Geophys. J. R. Astron. Soc. **13** (1967), no. 5, 529–539, DOI 10.1111/j.1365-246x.1967.tb02303.x.

[12] Antonin Chambolle, Stefano Lisini, and Luca Lussardi, *A remark on the anisotropic outer Minkowski content*, Adv. Calc. Var. **7** (2014), no. 2, 241–266, DOI 10.1515/acv-2013-0103. MR3187918

[13] Antonin Chambolle, Massimiliano Morini, and Marcello Ponsiglione, *Nonlocal curvature flows*, Arch. Ration. Mech. Anal. **218** (2015), no. 3, 1263–1329, DOI 10.1007/s00205-015-0880-z. MR3401008

[14] Ennio De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. Ital. (4) **1** (1968), 135–137 (Italian). MR0227827

[15] D. De Silva, O. Savin, and Y. Sire, *A one-phase problem for the fractional Laplacian: regularity of flat free boundaries*, Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), no. 1, 111–145. MR3234971

[16] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci, *A nonlocal free boundary problem*, SIAM J. Math. Anal. **47** (2015), no. 6, 4559–4605, DOI 10.1137/140999712. MR3427047

[17] Serena Dipierro and Enrico Valdinoci, *On a fractional harmonic replacement*, Discrete Contin. Dyn. Syst. **35** (2015), no. 8, 3377–3392, DOI 10.3934/dcds.2015.35.3377. MR3320130

[18] ———, *Continuity and density results for a one-phase nonlocal free boundary problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire, posted on In Press, DOI doi.org/10.1016/j.anihpc.2016.11.001.

[19] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, vol. 1991, Springer-Verlag, Berlin, 2010. Positivity preserving and nonlinear higher order elliptic equations in bounded domains. MR2667016

[20] Mariano Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983. MR717034

[21] ———, *Introduction to regularity theory for nonlinear elliptic systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993. MR1239172

[22] Enrico Giusti and Mario Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Un. Mat. Ital. (4) **1** (1968), 219–226 (Italian). MR0232265

[23] Rudolf Gorenflo, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer Monographs in Mathematics, Springer, Heidelberg, 2014. MR3244285

[24] Francesco Mainardi, *Fractional calculus and waves in linear viscoelasticity*, Imperial College Press, London, 2010. An introduction to mathematical models. MR2676137

[25] Connor Mooney and Ovidiu Savin, *Some singular minimizers in low dimensions in the calculus of variations*, Arch. Ration. Mech. Anal. **221** (2016), no. 1, 1–22, DOI 10.1007/s00205-015-0955-x. MR3483890

[26] Vladimír Šverák and Xiaodong Yan, *A singular minimizer of a smooth strongly convex functional in three dimensions*, Calc. Var. Partial Differential Equations **10** (2000), no. 3, 213–221, DOI 10.1007/s005260050151. MR1756327

[27] Vladimír Sverák and Xiaodong Yan, *Non-Lipschitz minimizers of smooth uniformly convex functionals*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 24, 15269–15276, DOI 10.1073/pnas.222494699. MR1946762

[28] Ray Yang, *Optimal regularity and nondegeneracy of a free boundary problem related to the fractional Laplacian*, Arch. Ration. Mech. Anal. **208** (2013), no. 3, 693–723, DOI 10.1007/s00205-013-0619-7. MR3048593

[29] José António Tenreiro Machado, Francesco Mainardi, Virginia Kiryakova, and Teodor Atanacković, *Fractional calculus: D'où venons-nous? Que sommes-nous? Où allons-nous?*, Fract. Calc. Appl. Anal. **19** (2016), no. 5, 1074–1104, DOI 10.1515/fca-2016-0059. MR3571004