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LINEAR CORRELATIONS OF THE k-TH DIVISOR FUNCTION

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1. Linear correlations of the Divisor Function

In the recent work of mine [3], I studied the analytic continuation of Dirichlet series of the form

$$\mathcal{D}(s_1,\dots,s_h,w_1,\dots,w_h) \sum_{n_1m_1+\cdots+n_hm_h=m_{h+1}n_{h+1}+\cdots+n_{h+r}m_{h+r}} \frac{1}{n_1^{s_1}\cdots n_{h+r}^{s_{h+r}}m_1^{w_1}\cdots m_{h+r}^{w_{h+r}}}$$

where $h,r\geq 1$, $h+r\geq 3$. This series converges to a holomorphic function provided that $\Re(w_i),\Re(s_i)>1-\frac{1}{h+r}$ for all i and I was able to prove that it admits meromorphic continuation to the larger domain $\Re(s_i),\Re(w_i)>1-\frac{1}{h+r}-\delta_{h+r}$ for some $\delta_{h+r}>0$. This result has several applications. For example, it was used:

- in the computation of the iterated 4-th moment of Dirichlet L-functions [2];
- to prove an asymptotic formula, with power saving error term, for
$$\sum_{\substack{n_1+\cdots n_h=n_{h+1}+\cdots+n_{h+r}\\ n_1\cdots n_{h+r}<B}} d(n_1)\cdots d(n_{h+r}),$$
as $B\to\infty$ and with $d(n)$ denoting the divisor function [3]
- to verify the Manin-Peyre conjecture on the Fano variety
$$x_1y_1+\cdots+x_hy_h=0$$
with $((x_1,\dots,x_h),(y_1,\dots,y_h))\in(\mathbb{P}^{h-1}(\mathbb{Q}))^2$ [3].

Furthermore, it gives (cf. [5] page 4) the full asymptotic formula, with power saving error term, for the number of solutions of height less than B on the threefold

$$\frac{x_1}{x_1}+\frac{x_2}{y_2}+\frac{x_3}{y_3}=0$$

when embedded as $((x_1,x_2,x_3),(y_1,y_2,y_3))\in(\mathbb{P}^2(\mathbb{Q}))^2$, $y_1y_2y_3\neq 0$. This improves upon the work [5], by Blomer, Brüdern and Salberger, where an asymptotic formula without power saving error term was obtained.

2. Linear correlations of the k-th Divisor Function

The project I intend to work on is that of generalizing the aforementioned work considering the higher level Dirichlet series

$$\mathcal{D}_k(s_{1,1},\dots,s_{k,1},\dots,s_{1,h},\dots,s_{k,h})=\sum_{n_{i,j}\geq 1}^{*}\prod_{\substack{1\leq i\leq k,\\ 1\leq j\leq h+r}} n_{i,j}^{-s_{i,j}},$$

where $\sum^*$ indicates that the summands are restricted to the solutions of the equation

$$n_{1,1}\cdots n_{k,1}+\cdots+n_{1,h}\cdots n_{\ell,h}=n_{1,h+1}\cdots n_{\ell,h+1}+\cdots+n_{1,h+r}\cdots n_{\ell,h+r}.$$

It is not difficult to show that this series converges provided that $\Re(s_{i,j})>1-\frac{1}{h+r}$ and as before, one seeks to give a meromorphic extension to a wider range. The work of Blomer and Brüdern [4] provides the tools needed to deal with certain ranges of the variables, but there are some non negligible "unbalanced" ranges where one needs new non-trivial bounds.

The meromorphic continuation of $\mathcal{D}_k$ would have several applications. Indeed, it would readily imply an asymptotic formula with power-saving error term for the following correlation sum for the k-th divisor function:

$$\sum_{\substack{n_1+\cdots n_h=n_{h+1}+\cdots+n_{h+r}\\ n_1\cdots n_{h+r}<B}} d_k(n_1)\cdots d_k(n_{h+r}),$$

where $d_k(n)$ is the k-th divisor function and as $B\to\infty$. Notice that this differs from the work [4] for the different order of counting; in particular, we remark that obtaining analytic continuation in each variable $s_{i,j}$ in $\mathcal{D}$ allows for great flexibility in the way the solutions are ordered, a flexibility which is often needed by several applications. Indeed, for example this flexibility would allow to deal also with several embeddings of the variety

$$\frac{x_1}{y_1}+\cdots+\frac{x_\ell}{y_\ell}. \tag{2.1}$$

In [7] Destagnol considered the embedding of this equation in $\mathbb{P}^{2\ell-1}(\mathbb{Q})$ obtaining the asymptotic for the number of solutions of bounded height, generalizing the work [6] of Blomer Brüdern and Selberger who considered the case $\ell=3$. The meromorphic extension for $\mathcal{D}_k$ would allow for an improvement over Destagnol's result, giving the full main term with power saving error term. It would also allow to consider the embedding of (2.1) in $(\mathbb{P}^{\ell-1}(\mathbb{Q}))^2$, obtaining the full main term also in this case thus extending the work [5] where the asymptotic was obtained in the case $\ell=3$.

3. Work Package and Deliverables

If I were to be funded, my plan would be to start working on the project in January 2018 after having hired a postdoc with the funding allocated. We'd first start with providing meromorphic continuation for the series $\mathcal{D}_k$ and then move to proving the various applications.

Part of the funding would be used to attend conferences and workshops in Analytic Number Theory and some would be used to invite expert in the field to visit the University of Genoa.

I expect that my work [3] combined with the ideas from [5] would give the desired meromorphic continuation for $\mathcal{D}_k$. Once established this, I'm confident I will be able to obtain the various aforementioned applications as the work needed is similar in spirit to several of my earlier works such as, in particular, [3], [2] and [1]. Thus, the goal and expectation is that of publishing two research papers, one with the theorem on $\mathcal{D}_k$ and one on the applications.

References

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