FINAL REPORT

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Following is the summary of my research activities from 1st December 2015 to present. In this period I have

- worked on *Artinian level algebras of socle degree 4* with M. E. Rossi [MR17];
- worked on *The structure of the inverse system of level K-algebras* with L. Tozzo [MT17];
- worked on *A filtration of the Sally module and the first normal Hilbert coefficient* with K. Ozeki and M. E. Rossi [MOR17];
- finished my earlier projects with K. Saloni [MS16] and C. D'Cruz [DM16];
- revised my paper with P. Chakraborty [CM16];
- had scientific visits and delivered talks (see Sections 7 and 8 for details);
- refereed a paper for Journal of Pure and Applied Algebra;
- reviewed papers for mathSciNet and zbMATH (see Section 9);
- delivered series of lectures (total 8 hours) on Macaulays inverse system at the department of mathematics, university of Genova.

In Section 2 we give a list of projects which are in progress and in Section 3 we describe our future plans.

1. Work Done

1.1. **Project with M. E. Rossi [MR17]:** In this project we characterized Hilbert functions of level rings of socle degree 4 and refined a result of Elias and Rossi [ER15].

Let $R = K[\![x_1, \dots, x_r]\!]$ be the formal power series ring in r variables where K is an algebraically closed field of characteristic zero. Let $A = R/I$ be an Artinian local ring. We set $\mathcal{M} = (x_1, \dots, x_r)$ and $\mathfrak{m} = \mathcal{M}/I$. Then $\operatorname{Soc}(A) = (0 :_A \mathfrak{m})$ is the socle of A. We denote by s the *socle degree* of A, that is the maximum integer j such that $\mathfrak{m}^j \neq 0$. The *type* of A is $\tau := \dim_K \operatorname{Soc}(A)$. Recall that A is said to be *level* of type τ if $\operatorname{Soc}(A) = \mathfrak{m}^s$ and $\dim_K \mathfrak{m}^s = \tau$. If A has type 1, equivalently $\dim_K(0 : \mathfrak{m}) = 1$, then A is *Gorenstein*.

Recall that the Hilbert function of A, $h_i = h_i(A) = \dim_K \mathfrak{m}^i/\mathfrak{m}^{i+1}$, is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(A) = \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. In [Mac27] Macaulay characterized the possible sequences h of positive integers h_i that can occur as the Hilbert function of A. Since then there has been a great interest in commutative algebra in determining the sequences of positive integers that can occur as the Hilbert function of A with additional properties (for example, complete intersection, Gorenstein, level, etc). A sequence of positive integers $h = (h_0, h_1, \dots, h_s)$ satisfying Macaulay's criterion, that is $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for $i = 1, \dots, s-1$, is called an *O-sequence*.

Oggetto: Re: Request
Data: mercoledì 15 novembre 2017 23:17:14 Ora standard dell'Europa centrale
Da: Shreedevi Masuti <masuti.shree@gmail.com>
A: Elisabetta Esposito <esposito@altamatematica.it>
Allegati: image002.png, image001.png, image003.png, INdAM_finalreport.pdf

Dear Esposito,
Please find the attached copy of my final report. Please inform me if you need any other documents.

Regards,
Shree.

On Mon, Nov 6, 2017 at 3:52 PM, Elisabetta Esposito <esposito@altamatematica.it> wrote:

> YES, YOUR FINAL REPORT: THE TITLE IS FINAL REPORT
>
> OK
>
> BYE EE
>
> *Dott.ssa Elisabetta Esposito*
>
> *Responsabile Ufficio Affari Generali*
>
> *e Segreteria della Presidenza*
>
> *Componente del Redress Committee dei Progetti INdAM Cofund e DP Cofund*
>
> *Istituto Nazionale di Alta Matematica "Francesco Severi" - INdAM*
>
> *Piazzale Aldo Moro, 5 Città Universitaria 00185 Roma*
>
> *Tel. 06/490320*
>
> *Fax 06/4462293*
>
> *Cell. 329/7241326*
>
> esposito@altamatematica.it esposito.altamatematica@pec.it http://www.altamatematica.it
>
> ---
>
> **Da:** Shreedevi Masuti <masuti.shree@gmail.com>

Data: lunedì 6 novembre 2017 10:47
A: Elisabetta Esposito <esposito@altamatematica.it>
Oggetto: Re: Request

Ok. sure, I will submit soon. This is a report of 2 years and not last 1 year, is it correct ?

Regards,

Shree.

On Mon, Nov 6, 2017 at 10:50 AM, Elisabetta Esposito <esposito@altamatematica.it> wrote:

> Yes, of course: your final report…as soon as possible,
>
> Best regard
>
> See you…
>
> elòisabetta
>
> iNδAM
>
> *Dott.ssa Elisabetta Esposito*
>
> *Responsabile Ufficio Affari Generali*
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> *e Segreteria della Presidenza*
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> *Componente del Redress Committee dei Progetti INdAM Cofund e DP Cofund*
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> esposito@altamatematica.it esposito.altamatematica@pec.it http://www.altamatematica.it

Da: Shreedevi Masuti <masuti.shree@gmail.com>
Data: venerdì 3 novembre 2017 15:33
A: Elisabetta Esposito <esposito@altamatematica.it>
Oggetto: Re: Request

Dear Esposito,

Thank you very much for your inspiring words. Do I need to submit a report ?

Regards,

Shreedevi.

On Fri, Nov 3, 2017 at 4:30 PM, Elisabetta Esposito <esposito@altamatematica.it> wrote:

> Dear Masuti, I am very happy for you because you are a very good fellow for us!
>
> I'm waiting for your final report.
>
> For tickets you can ask to feliciangeli@altamatematica.it
>
> All the best, EE
>
> INdAM
>
> *Dott.ssa Elisabetta Esposito*
>
> *Responsabile Ufficio Affari Generali*
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> *e Segreteria della Presidenza*
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Da: Shreedevi Masuti <masuti.shree@gmail.com>
Data: venerdì 3 novembre 2017 13:15
A: esposito <esposito@altamatematica.it>
Oggetto: Request

Dear Esposito,

I will finish my post doc position in November and will go back to India in the end of this month. It will be very helpful for me, if possible, I can get a salary for this month and the reimbursement amount by the end of this month (so that I can close my bank account in Italy).

I have booked my ticket to return to India. Is this fine if I ask for the reimbursement of this ticket now ? Please let me know so that I can apply for the reimbursement.

I am very thankful to you for all your help. I hope to meet you in future.

Regards,

Shreedevi.

A sequence $h = (1, h_1, \dots, h_s)$ is said to be a *level (resp. Gorenstein) O-sequence* if h is the Hilbert function of some level (resp. Gorenstein) K-algebra A. Characterization of level or Gorenstein O-sequences is a wide open problem in commutative algebra; see [Ber09, Theorem 2.6] for $h_1 = 2$ and [Ste14, Theorem 4.3] for $s \leq 3$. See [Sta78, Theorem 4.2] for $h_1 \leq 3$, [GHMY07], [Iar84, Theorem 4.6A] and [Iar04] for results in the graded setting, that is, R is a polynomial ring with the standard grading and I is a homogeneous ideal in R. One of the main results of this project is a characterization of local level O-sequences in the case $h_1 = 3$ and $s = 4$.

Theorem 1. Let $h = (1, 3, h_2, h_3, h_4)$ be an O-sequence.

(a) Let $h_4 = 1$. Then h is a Gorenstein O-sequence if and only if $h_3 \leq 3$ and $h_2 \leq \binom{h_3+1}{2} + (3 - h_3)$.

(b) Let $h_4 \geq 2$. Then h is a level O-sequence if and only if $h_3 \leq 3h_4$.

Theorem 1(a) is a consequence of the following more general result which holds for arbitrary h_1:

Theorem 2. (a) If $(1, h_1, \dots, h_{s-2}, h_{s-1}, 1)$ is a local Gorenstein O-sequence, then

$$h_{s-1} \leq h_1 \text{ and } h_{s-2} \leq \left(\binom{h_{s-1}+1}{2} + (h_1 - h_{s-1}) \right). \tag{1.1}$$

(b) If $h = (1, h_1, h_2, h_3, 1)$ is an O-sequence such that $h_2 \geq h_3$ and satisfies (1.1), then h is a local Gorenstein O-sequence.

Notice that Theorem 2(a) is a consequence of a more general result obtained by Iarrobino in [Iar94, Theorem 3.2A].

Recall that a local K-algebra A with the Hilbert function $(1, h_1, h_2, h_3, 1)$ is said to be *compressed* if $h_3 = h_1$ and $h_2 = \binom{h_1+1}{2}$. In [ER12, Theorem 3.3] and [ER15, Theorem 3.1] J. Elias and M. E. Rossi proved that if A is any compressed Gorenstein local K-algebra of socle degree $s \leq 4$, then A is *canonically graded* that is there exists a K-algebra isomorphism between A and its associated graded ring $gr_{\mathfrak{m}}(A)$. We remark that if $s = 3$ and A is any canonically graded Gorenstein local K-algebra then A is compressed. This statement no longer true for $s = 4$. However, we expect that if $s = 4$ and $h = (1, h_1, h_2, h_1, 1)$ is a Gorenstein O-sequence with $h_2 < \binom{h_1+1}{2}$, then there exists a Gorenstein K-algebra A with the Hilbert function h that is *not* canonically graded. We prove this result, provided h is unimodal ($h_2 \geq h_3$). We remark that to prove that a local K-algebra is not canonically graded is in general a very difficult task.

Theorem 3. Let $h = (1, h_1, h_2, h_3, 1)$ be a unimodal local Gorenstein O-sequence. Then every local Gorenstein K-algebra with the Hilbert function h is canonically graded if and only if $h_1 = h_3$ and $h_2 = \binom{h_1+1}{2}$.

1.2. **Project with L. Tozzo:** In this project we gave the structure of the inverse system of d-dimensional level K-algebras for any $d > 0$. This extends a recent result of Elias and Rossi [ER17].

Level rings were introduced by Stanley in [Sta77]. These rings are in between Cohen-Macaulay and Gorenstein rings. Since then they have been widely investigated, especially in the Artinian case. However, there are also many examples of level rings in positive dimension: Stanley-Reisner rings of matroid simplicial complexes, associated graded rings of semigroup rings corresponding to arithmetic sequences, determinantal rings corresponding to generic matrices or generic symmetric matrices.

Although several theory has been developed for level rings, they are not as well-understood as Gorenstein rings. One of the reasons for the lack of knowledge of level K-algebras of positive dimension is the absence of an effective method to construct level K-algebras. Macaulay's inverse system (see [Mac1916]) allows to construct level rings in the zero-dimensional case, as was shown by Emsalem [Ems78] and Iarrobino [Iar94]. Recently, Elias and Rossi [ER17] gave the structure of the inverse system of Gorenstein K-algebras of any dimension $d > 0$. In this project we extended their result and gave the structure of the inverse system of level K-algebras of positive dimension.

Recall that a homogeneous K-algebra A is *level* if the canonical module ω_A of A has a minimal set of generators of same degree. For the local case we take inspiration from [EI87], and say that a local K-algebra A is *level* if A/J is Artinian level for a general minimal reduction J of the maximal ideal. Notice that defining local level K-algebras of positive dimension is non-trivial. One of the reasons is that, if the associated graded ring of A is not Cohen-Macaulay, then Artinian reductions of A by minimal reductions of the maximal ideal may have different socle types.

Let $R = K[x_1, \dots, x_m]$ be a standard graded polynomial ring or $R = K[\![x_1, \dots, x_m]\!]$ a power series in m variables over the field K, and let I be an ideal of R (homogeneous if $R = K[x_1, \dots, x_m]$). It is well-known that the the injective hull of K is isomorphic to the *divided power ring* $\mathcal{D} := K_{DP}[X_1, \dots, X_m]$ as an R-module, where $\mathcal{D}$ has a structure of R-module via the contraction action. Therefore, the dual module $(R/I)^\vee := \operatorname{Hom}_R(R/I, E_R(K)) = \operatorname{Hom}_R(R/I, \mathcal{D})$ is isomorphic to an R-submodule of $\mathcal{D}$, called the *inverse system of* I and denoted by $I^\perp$. By Matlis duality it is clear that if R/I has positive Krull dimension, then $I^\perp$ is not finitely generated.

We investigate the structure of $I^\perp$ when R/I is a positive dimensional level K-algebra. In [ER17] the authors proved that $I^\perp$ is G_d*-admissible* if R/I is Gorenstein of dimension d and, vice versa, for any G_d-admissible submodule W of $\mathcal{D}$, $W^\vee$ is a d-dimensional Gorenstein K-algebra. In order to give the structure of the inverse system of level K-algebras we define L_d^τ*-admissible* R-submodule of $\mathcal{D}$ as follows:

Notation 1.2. For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_+^d$ we write $|\mathbf{n}| = n_1 + \cdots + n_d$. We set $\mathbf{e}_i := (0, \dots, 1, \dots, 0) \in \mathbb{N}^d$ (1 in the ith position) and $\mathbf{t}_d := (t, \dots, t) \in \mathbb{N}^d$ for $t \in \mathbb{N}$.

Definition 1.3. Let d and τ be positive integers. An R-submodule W of $\mathcal{D}$ is called *local* (resp. *graded*) L_d^τ*-admissible* if for every $(\overline{a_{ij}}) \in U$, $z_i := \sum_{j=1}^m a_{ij} x_j$ for $i = 1, \dots, d$ where U is some Zariski-dense open subset of K^{dm}, (resp. for some regular linear sequence) W admits a system of (resp. homogeneous) generators $\{H_{\mathbf{n}}^j : \mathbf{n} \in \mathbb{N}_+^d, j = 1, \dots, \tau\}$ with respect to $\underline{z} := z_1, \dots, z_d$ satisfying the following conditions:

(1) for all $\mathbf{n} \in \mathbb{N}_+^d$, $s_{\mathbf{n}} := \deg H_{\mathbf{n}}^1 = \deg H_{\mathbf{n}}^2 = \cdots = \deg H_{\mathbf{n}}^\tau$ and the forms of degree $s_{\mathbf{n}}$ of $H_{\mathbf{n}}^1, \dots, H_{\mathbf{n}}^\tau$ are linearly independent;

(2)

$$z_i \circ H_{\mathbf{n}}^j = \begin{cases} H_{\mathbf{n}-\mathbf{e}_i}^j & \text{if } \mathbf{n} - \mathbf{e}_i > \mathbf{0}_d \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

for all $\mathbf{n} \in \mathbb{N}_+^d$, $j = 1, \dots, \tau$ and $i = 1, \dots, d$;

(3) for all $i = 1, \dots, d$ and $\mathbf{n} \in \mathbb{N}_+^d$ such that $\mathbf{n} - \mathbf{e}_i > \mathbf{0}_d$

$$W_{\mathbf{n}} \cap V_{\mathbf{n}}^i \subseteq W_{\mathbf{n}-\mathbf{e}_i} \tag{1.5}$$

where $W_{\mathbf{n}} = \langle H_{\mathbf{n}}^j : j = 1, \dots, \tau \rangle$ and $V_{\mathbf{n}}^i := \langle Z_1^{k_1} \cdots Z_m^{k_m} : \mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}^m \text{ with } k_i \leq n_i - 2 \text{ and } |\mathbf{k}| \leq s_{\mathbf{n}} \rangle$.

The following theorem establishes a one-to-one correspondence between level K-algebras R/I of dimension $d > 0$ and L_d^τ-*admissible* submodules of $\mathcal{D}$.

Theorem 4. Let $R = K[\![x_1, \dots, x_m]\!]$ with $\operatorname{char}(K) = 0$ (resp. $R = K[x_1, \dots, x_m]$) and let $0 < d \le m$. There is a one-to-one correspondence between the following sets:
(1) d-dimensional local (resp. graded) level K-algebras of type τ;
(2) non-zero local (resp. graded) L_d^τ-admissible R-submodules $W = \langle H_{\mathbf{n}}^j : j = 1, \dots, \tau, \mathbf{n} \in \mathbb{N}_+^d \rangle$ of $\mathcal{D}$;

given by

$$\left\{ \begin{array}{c} I \subseteq R \text{ such that } R/I \text{ is a local} \\ \text{(resp. graded) level } K\text{-algebra of} \\ \text{dimension } d \text{ and type } \tau \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{non-zero local (resp. graded)} L_d^\tau\text{-admissible} \\ R\text{-submodule } W \text{ of } \mathcal{D} \end{array} \right\}$$

$$\begin{array}{ccc} R/I & \longrightarrow & I^\perp \\ R/\operatorname{Ann}_R(W) & \longleftarrow & W \end{array}$$

We remark that the conditions given for L_d^τ-admissibility are not merely the "union" of the conditions given for G_d-admissible submodules. This is a symptom of the intrinsic complexity of level K-algebras in positive dimension, and it is one of the reasons why constructing examples of level algebras is hard. For example, in the Artinian case, as a consequence of Macaulay's inverse system, it is known that the intersection of Gorenstein ideals of same socle degree is always level. We gave an example which illustrates that an analogous statement is no longer true in positive dimension. For this reason Theorem 4 is an important tool, as it gives an effective method to construct level K-algebras.

1.3. **Project with K. Ozeki and M. E. Rossi:** In this project we gave new insights on the structure of the Sally module. We apply these results characterizing the almost minimal value of the first Hilbert coefficient in the case of the normal filtration in an analytically unramified Cohen-Macaulay local ring.

Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field $R/\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal of R. Let $\overline{I}$ denote the *integral closure* of I. It is well-known that there are integers $\overline{\mathrm{e}}_i(I)$, called the *normal Hilbert coefficients* of I, such that for $n \gg 0$

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_0(I)\binom{n+d}{d} - \overline{\mathrm{e}}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{\mathrm{e}}_d(I).$$

Here $\ell_R(N)$ denotes, for an R-module N, the length of N. Since $R/\mathfrak{m}$ is infinite there exists a minimal reduction $J = (a_1, \dots, a_d)$ of I and, under our assumptions, there exists a positive integer r such that $\overline{I^{n+1}} = J\overline{I^n}$ for $n \ge r$. We set

$$\overline{\mathrm{r}}_J(I) := \min\{r \ge 0 \mid \overline{I^{n+1}} = J\overline{I^n} \text{ for all } n \ge r\}$$

for the *reduction number* of I with respect to J.

We recall that $\overline{\mathrm{e}}_1(I) \ge 0$, but the bound can be more precise. It is well known that

$$\overline{\mathrm{e}}_1(I) \ge \overline{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}).$$

Moreover, the equality holds true if and only if $\overline{I^{n+1}} = J^n\overline{I}$ for every $n \ge 0$ and for every minimal reduction J of I (that is $\overline{\mathrm{r}}_J(I) \le 1$). In this case the normal associated graded ring

$\overline{G}(I) := \oplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ of I is Cohen-Macaulay. Recently Corso, Polini and Rossi [CPR16] showed that if the equality $\bar{e}_1(I) = \bar{e}_0(I) - \ell_R(R/\overline{I}) + 1$ holds true, then depth $\overline{G}(I) \geq d-1$.

By [Ito92] it is known that

$$\bar{e}_1(I) \geq \bar{e}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I})$$

and the equality holds true if and only if $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ for every $n \geq 1$ (that is $\bar{r}_J(I) \leq 2$). In this case the normal associated graded ring $\overline{G}(I)$ of I is Cohen-Macaulay. We notice that $\ell_R(\overline{I^2}/J\overline{I})$ does not depend on a minimal reduction J of I (see for instance [RV10])

Thus the ideals I with $\bar{e}_1(I) = \bar{e}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I})$ enjoy nice properties and it seems natural to ask when the equality $\bar{e}_1(I) = \bar{e}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$ holds. The main purpose of this project is to explore this equality. The following theorem presents the structure of the Sally module in this case.

Theorem 5. Let $(R,\mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d>0$ and I an $\mathfrak{m}$-primary ideal in R. Then following statements are equivalent:

(1) $\bar{e}_1(I) = \bar{e}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$;
(2) $\overline{C} \simeq B(-m)$ as graded T-modules for some $m \geq 2$;
(3) $\ell_R(\overline{I^{m+1}}/J\overline{I^m}) = 1$ and $\overline{I^{n+1}} = J\overline{I^n}$ for all $2 \leq n \leq m-1$ and $n \geq m+1$ for some $m \geq 2$.

In this case, the following assertions follow:

(i) $\bar{r}_J(I) = m+1$.
(ii) $\mathrm{HS}_{\overline{G}(I)}(z) = \frac{\ell_R(R/I)+(e_0(I)-\ell_R(R/\overline{I})-\ell_R(\overline{I^2}/J\overline{I}))z+\ell_R(\overline{I^2}/J\overline{I})z^2-z^m+z^{m+1}}{(1-z)^d}$.
(iii) $\bar{e}_2(I) = \ell_R(\overline{I^2}/J\overline{I}) + m$ and $\bar{e}_i(I) = \binom{m}{i-1}$ for $3 \leq i \leq d$.
(iv) depth $\overline{G}(I) \geq d-1$.
(v) $\overline{G}(I)$ is Cohen-Macaulay if and only if $\overline{I^3} \nsubseteq J$. In this case, we have $m=2$.

We remark that if I is integrally closed, the corresponding equality was studied in [OR16]. In [OR16] the authors proved that if the equality $e_1(I) = e_0(I) - \ell_R(R/I) + \ell_R(I^2/JI) + 1$ holds true, then the depth of the associated graded ring $G(I) := \bigoplus_{n\geq 0} I^n/I^{n+1}$ can be any integer between 0 and $d-1$. This "bad" behavior motives our study in the case of the normal filtration proving that it enjoys nice properties as compared to the I-adic filtration.

Moreover, we deduce some consequences of Theorem 5 which include the already quoted result [CPR16]. We also gave an example which illustrates that depth $\overline{G}(I)$ in Theorem 5 can not be improved.

2. Work Under Progress

2.1. **With J. Jelisiejew and M. E. Rossi:** The main aim of this project is to understand the Hilbert function of local complete intersections $A = R/(f,g,h)$ where $R = K[\![x,y,z]\!]$ and order f = order g = order $h = 2$.

By assumption A has the Hilbert function of the form $h = (1,3,3,h_3,\ldots,h_s = 1)$. By Macaulay's bound it is clear that $h_3 \leq 4$. In the following theorem we prove that if $h_3 \leq 3$, then every O-sequence is admissible for a complete intersection.

Theorem 6. Let

$$h := \left(1,3,3,\underbrace{3,\ldots,3}_{s},\underbrace{2,\ldots,2}_{t},\underbrace{1\ldots1}_{u},1\right)$$

be an O-sequence. Then there exists a complete intersection $A = R/(f,g,h)$ with the Hilbert function h. More precisely,

(a) (trivial case) Suppose that $s = t = u = 0$. Then $\operatorname{Ann}_R(XYZ) = (x^2, y^2, z^2)$ is a complete intersection with the Hilbert function $h = (1,3,3,1)$.

(b) Suppose that at least one of the s,t,u is non-zero. Then

$$\operatorname{Ann}_R(F) = \left(x^{s+t+u+2} - yz, y^{s+t+2} - xz, z^{s+2} - xy\right)$$

where $F := X^{s+t+u+3} + Y^{s+t+3} + Z^{s+3} + XYZ$ is a complete intersection with the Hilbert function h.

We have partial result For $h_3 = 4$. In the following theorem we give a necessary condition for an O-sequence to be admissible for a complete intersection.

Theorem 7. Suppose that $h = (1,3,3,4,h_4,\ldots,h_s = 1)$ is admissible for a complete intersection. Then $\Delta(h) \leq 2$ where $\Delta(h) := \max\{|h_i - h_{i-1}| : i = 1,\ldots,s\}$ and there exists at most one i for which $|h_i - h_{i-1}| = 2$.

More generally, we would like to understand the possible Hilbert functions of the form $h = (1,3,3,4,h_4,\ldots,h_s = 1)$ that are admissible for Gorenstein rings. We conjecture that:

Conjecture 2.1. If $h = (1,3,3,4,h_4,\ldots,h_s = 1)$ is a Gorenstein O-sequence, then h is admissible for a complete intersection.

2.2. **With K. Ozeki and M. E. Rossi:** The main goal of this project is to explore the structure of the Sally module in the case the second normal Hilbert coefficient of an $\mathfrak{m}$-primary ideal attains almost minimal value in an analytically unramified Cohen-Macaulay local ring.

Let $(R,\mathfrak{m})$ is an analytically unramified Cohen-Macaulay local ring of dimension $d > 1$ with infinite residue field, I is an $\mathfrak{m}$-primary ideal of R and $J \subseteq I$ is a minimal reduction of I. Recall that $\overline{\mathrm{e}}_2(I) \geq 0$. In fact, by [Ito92, Theorem 2(2)]

$$\overline{e}_2(I) \geq \overline{e}_1(I) - \overline{e}_0(I) + \ell_R(R/\overline{I}). \tag{2.2}$$

Moreover, the equality holds if and only if $\overline{I^{n+2}} = I^n\overline{I^2}$ for all $n \geq 0$. In this case the normal associated graded ring $\overline{G}(I)$ is Cohen-Macaulay. We are interested in the equality $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$. We expect that in this case also depth $\overline{G}(I) \geq d - 1$:

Conjecture 2.3. Let $(R,\mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring and I-an $\mathfrak{m}$-primary ideal in R. Suppose that $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$. Then depth $\overline{G}(I) \geq d - 1$.

In the following theorem we obtained the structure of Sally module in dimension 2. We remark that since depth $\overline{G}(I) \geq 1$, the Conjecture 2.3 is automatically satisfied for $d = 2$ (without any assumption on the coefficients). The following theorem gives more refined information when $\overline{\mathrm{e}}_2(I)$ attains almost minimal value.

Theorem 8. Let $(R,\mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension 2, I an $\mathfrak{m}$-primary ideal of R and $J \subseteq I$ a minimal reduction of I. Suppose that $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$. Then $\overline{C} \simeq B(-2)$. In particular, $\overline{r}(I) = 3$.

We observed that if the Itoh's conjecture [Ito92, page 116] is true, then the Conjecture 2.3 holds true. Since the Itoh's conjecture is still open, we do not know if the Conjecture 2.3 is true.

2.3. **With L. Brustenga:** This is my Pragmatic project with Laura Brustenga under the supervision of Enrico Carlini. The aim of this project is to compute the Waring rank of certain binary forms.

Let $S = \mathbb{C}[x, y]$ and $F \in S_d$. Recall that the *Waring rank* of F, denoted as $\operatorname{rk}(F)$ is

$$\operatorname{rk}(F) := \min\{r : F = \sum_{i=1}^{r} \lambda_i L_i^d \text{ where } L_i \in S_1 \text{ and } \lambda_i \in \mathbb{C}\}.$$

The long outstanding problem of finding the number of summands in a minimal Waring expansion of the generic form of degree d was solved, after being open for almost 100 years, by J. Alexander and A. Hirschowitz ([AH95]). Of course, solving the problem for the generic form of degree d does not always give information about any specific form of degree d, and the problem of finding the length of the minimal Waring expansion for specific forms has also been a continuing source of interesting speculations and lovely results. It is interesting to note that although the Waring problem is a very interesting and stimulating problem in purely algebraic terms, it has a surprising number of intimate connections with problems in areas as seemingly disparate as algebraic geometry and communication theory.

It is well-known that $\operatorname{rk}(F) \leq d$ and due to Sylvester it is known that $\operatorname{rk}(xy^{d-1}) = d$. Recently, E. Carlini, M. Catalisano and A. Geramita proved that $\operatorname{rk}(x^a y^b) = b + 1$ where $1 \leq a \leq b$ [1] [CCG12]. Our goal is to find the Waring rank of binomial forms and square-free binary forms. We proved that:

Theorem 9. Let $F = x^r y^{s+\alpha} + a x^{r+\alpha} y^s$ where $a \neq 0$, $1 \leq \alpha \leq r$.

(a) If $s \geq r + \alpha$. Then $\operatorname{rk}(F) = s + 1$.
(b) Suppose $s < r + \alpha$. Set $\delta := r + \alpha - s$. Then

$$\operatorname{rk}(F) = \begin{cases} s+2 & \text{if } r \equiv 0 \mod \alpha \text{ where } 1 \leq \lceil \frac{\delta-1}{2} \rceil \text{ and } r = s \\ r+\alpha-j & \text{if } r \equiv j \mod \alpha \text{ where } 1 \leq j \leq \lceil \frac{\delta-1}{2} \rceil - 1, \text{ OR } j = 0 \text{ and } r < s \\ s+j+1 & \text{if } r \equiv j \mod \alpha \text{ when } \delta \text{ is odd and } j = \frac{\delta-1}{2} \\ s+j & \text{if } r \equiv \delta - j \mod \alpha \text{ where } 1 < j \leq \lceil \frac{\delta-1}{2} \rceil \\ s+1 & \text{if } r \equiv j \mod \alpha \text{ where } \delta - 1 \leq j \leq \delta \\ r+\alpha+1 & \text{if } r \equiv j \mod \alpha \text{ where } \delta + 1 \leq j \leq \alpha - 1 \end{cases}$$

In particular, $\operatorname{rk}(F)$ does not depend on a.

Theorem 10. There are exactly $\binom{d-1}{2}$ square-free binary forms of degree d and Waring rank 2.

3. Future Plans

3.1. **Based on work with M. E. Rossi:** In Theorem 2(b) we gave a characterization of unimodal Gorenstein sequences of socle degree 4 in any codimension. It will be interesting to understand Gorenstein O-sequences of socle degree 5.

Problem 3.1. Find a characterization of Gorenstein O-sequences of socle degree 5 which are unimodal in any codimension.

We would like to remark that this problem is not a straightforward generalization of Theorem 2(b). In fact, $h = (1, 3, 3, 4, 3, 1)$ satisfies the condition (1.1), but is not a Gorenstein sequence.

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[1] In [CCG12] authors obtained a formula for the Waring rank of monomials in arbitrary number of variables. In the case of 2 variables their formula also follows from Sylvester's algorithm.

Question 3.2. Which of the Gorenstein sequences in Theorem 2(b) are admissible for local complete intersections ?

3.2. **Based on work with L. Tozzo:** In [MT17] we gave the structure of the inverse system of level rings of any dimension $d > 0$. Inspired by this we ask:

Problem 3.3. (a) Find the structure of the inverse system of distinct points in $\mathbb{P}^{n-1}_K$ where $n \geq 2$. We remark that distinct points in $\mathbb{P}^{n-1}_K$ corresponds to a reduced ring $A = R/I$ of dimension one where $R = K[x_1, \ldots, x_n]$. Since A is reduced, A is Cohen-Macaulay. By [MT17, Remark 4.15], we know the structure of the inverse system of Cohen-Macaulay rings of any dimension $d > 0$. In this problem we are interested in finding the inverse system of **reduced** (Cohen-Macaulay) ring of dimension one.

(b) Find the structure of the inverse system of Gorenstein **domains** of any dimension $d > 0$.

4. Papers communicated

- **S. K. Masuti**, K. Ozeki and M. E. Rossi, *A filtration of the Sally module and the first normal Hilbert coefficient*, submitted to J. Algebra in a special volume dedicated to Craig Huneke (2017).
- **S. K. Masuti** and L. Tozzo, *The structure of the inverse system of level K-algebras*, (2017), available at `arXiv:1708.01800`.
- **S. K. Masuti** and M. E. Rossi, *Artinian level algebras of socle degree* 4, (2017), available at `arXiv:1701.03180`.
- C. D'Cruz and **S. K. Masuti**, *Cohen-Macaulayness and Gorensteinness of Symbolic Blowup algebras of certain monomial curves*, (2016), available at `arXiv:1610.03658`.

5. Paper accepted

- **S. K. Masuti** and K. Saloni, *On the Finiteness of the set of Hilbert Coefficients*, to appear in Journal of Commutative Algebra.

6. Paper Revised

- P. Chakraborty and **S. K. Masuti**, *Rational homotopy of maps between certain complex Grassmann manifolds*, to appear in Mathematica Slovaca, available at `arXiv:1504.07362`.

7. Scientific Visits

- Participated in a conference on **Instruments of Algebraic Geometry**, University of Bucharest, Bucharest, Romania, September 18-September 22, 2017.
- Participated in a school on **School and Workshop on Syzygies**, Fondazione Bruno Kessler-IRST, Trento, Italy, September 4-September 9, 2017.
- Participated in a conference on **The Prospects for Commutative Algebra**, Hotel Nikko Osaka, Osaka, Japan, July 10-July 14, 2017.
- Participated in **PRAGMATIC on Powers of Ideals and Ideals of Powers**, Università di Catania, Catania, Italy, June 19-July 7, 2017.
- Visited the Institute of Mathematics, University of Osnabrück, Osnabrück, Germany from 5-12 November, 2016 (this visit was a part of Vigoni Project).
- Visited Chennai Mathematical Institute, Chennai, India, from 29th August 2016 to 2nd September 2016. During this visit I worked on a project with Prof. Manoj Kummini (this project is still going on) and Prof. Clare D'Cruz (this project is finished [DM16]).

8. Talks Delivered

- On *symbolic Rees algebras of certain monomial curves* in the conference on **The Prospects for Commutative Algebra**, Hotel Nikko Osaka, Osaka, Japan, July 10-July 14, 2017.
- Gave a seminar on *Symbolic Rees algebras of certain monomial curves* at **IIT Bombay**, Mumbai, India, on 28th February 2017.
- Gave a seminar on *Hilbert functions of graded and local algebras* at **Harish-Chandra Research Institute**, Allahabad, India, on 20th February 2017.
- On *Symbolic Blowup algebras of certain monomial curves* at the Institute of Mathematics, University of Osnabrück, Osnabrück, Germany, on 8th November 2016.
- On *Maps between certain complex Grassmann manifolds* at the Department of Mathematics, University of Genova, Genova, Italy, on 23rd March 2016.

9. Papers Reviewed

9.1. For MathSciNet:.

- Y. Irani, *Cominimaxness with respect to ideals of dimension one*, Bull. Korean Math. Soc. **54** (2017), no. 1, 289-298.
- M. Aghapournahr and K. Bahmanpour *Cofiniteness of general local cohomology modules for small dimensions*, Bull. Korean Math. Soc. **53** (2016), no. 5, 1341-1352.
- V. Van Lierde, *Quasi-one-fibered and projectively full ideals in 2-dimensional Muhly rational singularities*, Eur. J. Math. **2** (2016), no. 3, 798-808.
- J. Elias and R. Homs, *On the analytic type of Artin algebras*, Comm. Algebra **44** (2016), no. 6, 2277-2304.
- R. Burity, A. Simis and S. Tohăneanu, *On a conjecture of Vasconcelos via Sylvester forms*, J. Symbolic Comput. **77** (2016), 39-62.
- E. K. Sabine, A. V. Jayanthan and H. Srinivasan, *On the number of generators of ideals defining Gorenstein Artin algebras with Hilbert function* $(1, n+1, 1+\binom{n+1}{2}, \ldots, \binom{n+1}{2}+1, n+1, 1)$, Beitr. Algebra Geom. **57** (2016), no. 1, 173-187.
- P. Görlach, C. Riener and T. Weißer, *Deciding positivity of multisymmetric polynomials*, J. Symbolic Comput. **74** (2016), 603-616.
- D. Ghosh, *Asymptotic linear bounds of Castelnuovo-Mumford regularity in multigraded modules*, J. Algebra **445** (2016), 103-114.
- K. F. E. Chong, *Hilbert functions of colored quotient rings and a generalization of the Clements-Lindstrm theorem*, J. Algebraic Combin. **42** (2015), no. 1, 1-23.
- N. T. Cuong, S. Goto and H. N. Van, *On the cofiniteness of generalized local cohomology modules*, Kyoto J. Math. **55** (2015), no. 1, 169-185.
- M. Jahangiri, N. Shirmohammadi and Sh. Tahamtan, *Tameness and Artinianness of graded generalized local cohomology modules*, Algebra Colloq. **22** (2015), no. 1, 131-146.
- I. Swanson, *Integral closure*, Commutative algebra, 331-351, Springer, New York, 2014.

9.2. For zbMATH:.

- M. Sedghi, *Ratliff-Rush closures and linear growth of primary decompositions of ideals*, J. Algebra Appl. **16** (2017).
- N. Botbol, L. Busé, M. Chardin, S. H. Hassanzadeh, A. Simis and Q. H. Tran, *Effective criteria for bigraded birational maps*, J. Symb. Comput. **81** (2017), 69-87.

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[CM16] P. Chakraborty and S. K. Masuti, *Rational homotopy of maps between certain complex Grassmann manifolds*, to appear in Mathematica Slovaca, available at `arXiv:1504.07362`. 1

[CPR16] A. Corso, C. Polini and M. E. Rossi, *Bounds on the normal Hilbert coefficients*, Proc. Amer. Math. Soc. **144** (2016), 1919–1930. 5

[DM16] C. D'Cruz and S. K. Masuti, *Cohen-Macaulayness and Gorensteinness of Symbolic Blowup algebras of certain monomial curves*, available at `arXiv:1610.03658`. 1, 8

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[MT17] S. K. Masuti and L. Tozzo, *The structure of the inverse system of level K-algebras*, communicated (2017), available at `arXiv:1708.01800`. 1, 8

[MR17] S. K. Masuti and M. E. Rossi, *Artinian level algebras of socle degree* 4, communicated (2017), available at `arXiv:1701.03180`. 1

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