Report on research activities.

COFUND-2012 INdAM Incoming Fellowships, cofunded by Marie Curie actions.

Research program:

"Nonlinear Dispersive Equations in Semiclassical settings and Product spaces" (NLDISC)

Period: 01/12/2015-31/08/2016

Location: Dipartimento di Matematica, Università di Pisa.

During my position at Università of Pisa, financed by the INdAM incoming fellowship, my research activities were about the qualitative study of nonlinear dispersive PDEs. I was focusing on two different problems:

I Investigate Strichartz estimates for potentials usually used in systems of semiclassical equations, dealing with crossing phenomena.

II Develop the scattering theory for The Nonlinear Klein-Gordon Equation and understand the long time behaviour of the solutions in product spaces.

I will briefly resume each point of the project, by describing previous results that have motivated the studies, before giving details about the results I obtained.

1   First topic: Scattering theory for nonlinear Klein-Gordon equations in product spaces

The main motivation is to investigate properties inherited from the linear part in some unusual mixed settings. In other words, we would like to develop the scattering theory for different PDEs and understand the behaviour of the solutions in some modified geometry.

1.1   Quick review and presentation of the problem

The case of Schrödinger equation has already been treated by Tzvetkov-Visciglia, considering

$$i\partial_t u - \Delta_{x,y} u = \pm|u|^2 u, \quad u(0, x, y) = f(x, y), \quad (t, x, y) \in \mathbf{R} \times \mathbf{R}^d \times \mathcal{M}^k,$$

where $\mathcal{M}^k$ is a compact Riemannian manifold of dimension k. For a H^1-subcritical nonlinearity, scattering or non-scattering properties are known on euclidean spaces on one hand and on compact manifolds on the other hand:

- On euclidean spaces, for small data and sufficiently small nonlinearity, scattering is expected.

- On compact Riemannian manifolds, scattering not expected, even for small data.

An interesting question is: what happens if we "mix" the spaces, namely, considering the Cauchy problem on $\mathbf{R}^d \times \mathcal{M}^k$, for small data, is the dispersive nature of the euclidean part sufficient to have scattering ? The answer "yes" is given in [34, 35], for small data first, and then for large data in the energy space, for the mixed torus $\mathbf{R}^d \times \mathbf{T}$ and for a nonlinearity which is energy-subcritical for considering the total dimension $(d + 1)$. Other linked results are discussed in [19] for product spaces, and in [1, 4] for partial confinement, a related problem.

For mathematical and physical interest, since Wave/Klein-Gordon equations are also dispersive, we would like to investigate the validity of same kind of results for $(t, x, y) \in \mathbf{R} \times \mathbf{R}^d \times \mathcal{M}^k$

$$i\partial_t u - \Delta_{x,y} u + m^2 u = \pm|u|^2 u, \quad u(0, x, y) = f(x, y), \ \dot{u}(0, x, y) = g(x, y), \quad (1)$$

where $m = 0$ (Wave) or $m \neq 0$ (Klein-Gordon). Previous works for Klein-Gordon equation in euclidean cases have been mainly performed by Brenner [3], Ginibre-Velo [12, 13], Pecher [30, 31], and completed in several papers of Ibrahim-Masmoudi-Nakanishi (see [24] for example). Results of existence in "compact" settings have been proved by Delort, Delort-Szeftel ([7, 9, 6, 8]), Fang-Zhang ([10]) and in various settings (with potentials and/or in other type of space structures) by Zhang, Imekraz...

1.2   Strategy and preliminary estimates

The basic idea to face the objective is to adapt the arguments given in [34, 35] for the Schrödinger equation: the main tool to prove scattering results is global in time Strichartz estimates for $\mathbf{R}^d \times \mathcal{M}^k$, using known information from the euclidean case, and deducing suitable estimates, minimizing (or even better, avoiding) loss of regularity, for the whole

space. The main difficulty to adapt previous ideas directly is the following point : one can use the decomposition of all functions involved in (1) on $\mathcal{M}^k$, using $-\Delta_y \Phi_j = \lambda_j \Phi_j$

$$u(t,x,y) = \sum_j u_j(t,x)\Phi_j(y), \quad (t,x,y) \in \mathbf{R} \times \mathbf{R}^d \times \mathcal{M}^k$$

and for each j, (except in the case $\lambda_j = m = 0$ which requires a deeper study) one has a "flat" Klein-Gordon equation on $\mathbf{R} \times \mathbf{R}^d$

$$\partial_t^2 u_j - \Delta_x u_j + (\lambda_j + m^2)u_j = F_j \quad ; \quad u_j(0,.) = f_j, \ \dot{u}_j(0,.) = g_j.$$

Here, usual Strichartz estimates are available (see [29, 12, 13, 25]) in some Sobolev/Besov spaces for $\lambda_j + m^2 = 1$. Defining admissibility for Klein-Gordon equations:

Definition 1.1. *A pair (p,q) is* **admissible** *if $2 \le q \le \frac{2d}{d-2}$ ($2 \le q \le \infty$ if $d = 1$, $2 \le q < \infty$ if $d = 2$) and*

$$\frac{2}{p} = \delta(q) := d\left(\frac{1}{2} - \frac{1}{q}\right).$$

and an exponent that will be used in the Besov spaces:

Notation. Consider (p,q) an admissible pair. We then denote by s the following exponent:

$$s = 1 - \frac{1}{2}\left(\frac{d}{2}+1\right)\cdot\left(\frac{1}{q'}-\frac{1}{q}\right) = 1 - \frac{1}{2}\left(\frac{d}{2}+1\right)\cdot\left(1-\frac{2}{q}\right). \tag{2}$$

one has

Proposition 1.2 ("Flat" Strichartz with Besov Spaces)**.** *Let $d \ge 1$ and consider u the solution to (1) on $\mathbf{R}^d$ with $m^2 = 1$. Consider two admissible pairs $(p_1,q_1),(p_2,q_2)$, s_1,s_2 as in (2). Then*

$$\|u\|_{L^{p_1}B_{q_1,2}^{s_1}} \lesssim \|f\|_{H^1} + \|g\|_{L^2} + \|F\|_{L^{p_2'}B_{q_2',2}^{1-s_2}}.$$

In particular, for $(p_2,q_2) = (\infty,2)$, one has

$$\|u\|_{L^{p_1}B_{q_1,2}^{s_1}} \lesssim \|f\|_{H^1} + \|g\|_{L^2} + \underbrace{\|F\|_{L^1 B_{2,2}^0}}_{\|F\|_{L^1 L^2}}.$$

Remark 1.3. We make two remarks here: first, it is possible to state the previous estimates for more general Besov spaces of type $B_{q,r}^s$. But $r = 2$ is sufficient here. The second remark deals with the data (f,g). Once again, one can state more general estimates for more general spaces H^s, H^{s-1}.

Our aim is to use a scaling argument to obtain inequalities for $\lambda_j + m^2 \neq 1$. Therefore, inhomogeneous Besov spaces are not easy to work with. Using embeddings between Besov and Sobolev spaces (given in books of Triebel or Adams), one can obtain "*exotic*" Strichartz estimates with non-optimal Lebesgue-type spaces from Proposition 1.2

$$\|u_j\|_{L_t^p L_x^q} \leq C(\lambda_j) \left(\|f_j\|_{H_x^1} + \|g_j\|_{L_x^2} + \|F_j\|_{L_t^\rho L_x^\theta} \right),$$

where $\max(\rho, \theta) \leq 2 \leq \min(p, q)$ with specific links depending on the equation between the exponents.

In [34, 35], the constant is independent of λ_j, thus, summing in j, the authors obtain "complete" Strichartz estimates on the whole product spaces, without any loss in y. Unfortunely for (1), estimates for each j will depend on λ_j, therefore, summing in j would not be as easy as for Schrödinger and requires a fine computation to minimize the loss in y. In fact, a big loss in the $y-$variable would be an issue to prove scattering for large data in the energy space, since the data g should not lose derivatives. Let us also remark that since one needs global in time Strichartz estimates, it is not possible to use Strichartz estimates on small time interval and see the "bad" term as a perturbation/source term in the wave equation.

1.3   Some results

With Prof N. Visciglia, we were able to prove some Strichartz estimates for product spaces without involving a loss of derivative on the right handside term (inhomogeneous estimate). The key point is to use simple $L_t^1 L_{x,y}^2$ norms for the source term but play with Sobolev embeddings to obtain what we need on the left handside. As a first try, the cubic NLKG can be handled with

$$\|u\|_{L_t^3 L_{x,y}^6} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_{x,y}^2} \right]$$

on $\mathbf{R}^3 \times \mathcal{M}$ or $\mathbf{R}^2 \times \mathcal{M}^2$ (H^1 critical cases for the whole space). These estimates were the basis to find more general ones for general power type nonlinaerities $|u|^{p-1}u$ on $\mathbf{R}^d \times \mathcal{M}^1$ or $\mathbf{R}^d \times \mathcal{M}^2$. In fact, one can obtain some estimates of the form

$$\|u\|_{L_t^p L_{x,y}^{2p}} \leq C \left[\|f\|_{H_{x,y}^1} + \|g\|_{L_{x,y}^2} + \|F\|_{L_t^1 L_{x,y}^2} \right]$$

for appropriate p (up to the $H^1(\mathbf{R}^d \times \mathcal{M}^k)-$critical exponent).
Focusing only on the critical case, we have proved:

Theorem 1.4. *Let $1 \leq d \leq 4$ and $\mathcal{M}^2$ be a 2-dimensional compact manifold. Then there exists $\delta > 0$ such that the Cauchy problem* (1) *has a unique global solution*

$$u \in C^0(\mathbf{R}, H^1) \cap C^1(\mathbf{R}, L^2) \cap L^{1+\frac{4}{d}}(\mathbf{R}, L^{2+\frac{8}{d}}),$$

Moreover the nonlinear solution scatters, in the following sense:

$$\forall (f,g) \in H^1 \times L^2 \quad \exists (f^\pm, g^\pm) \in H^1 \times L^2 \ s. \ t.$$
$$\lim_{t \to \pm\infty} \|u_\pm(t,x) - u(t,x)\|_{H^1} + \|\partial_t u_\pm(t,x) - \partial_t u(t,x)\|_{L^2} = 0$$

where $u(t,x)$ solves (1) *and*

$$\begin{cases} \partial_t^2 u_\pm - \Delta_{\mathbf{R}^d \times \mathcal{M}^2} u_\pm + u_\pm = 0, \\ \quad u_\pm(0,.) = f_\pm \in H^1, \quad \partial_t u_\pm(0,.) = g_\pm \in L^2. \end{cases}$$

Note that the theorem is valid for subcritical cases, such that $p \geq 1 + 4/d$, which is the $L^2(\mathbf{R}^d)-$critical exponent.

The result was submitted during the aforementioned period and published later in Communications in Contemporary Mathematics ([23]).

The next step was to deal with large data theory, involving much more technical tools (either in the H^1-subcritical or critical case). The subcritical case was discussed with L. Forcella of the Scuola Normale Superiore, Pisa. We adapted a concentration-compactness and rigidity method, using Morawetz/Nakanishi estimates. Some of the arguments were written under the postdoc position, funded by INdAM-Cofund. In particular, the Morawetz/Nakanishi estimate, valid for the one-dimensional equation, the profile decomposition - adapted to product spaces, were already investigated during that period. The whole work was finished in 2017, and is available as a preprint version (see [11]).

2   Second topic: Propagation of semiclassical wave packets

2.1   Known results and open problems

We first focus on nonlinear Schrödinger equations which are widely used to describe Bose-Einstein condensates, modelling a new state of the matter with interesting physical properties ([16, 17, 18]). It is usually described in a trapped condition or under confinement - a suitable potential V (usually the harmonic potential is used in experiments) - to study its properties, through Gross-Pitaevskii equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2M} \Delta_x + V(x) + g|\psi|^2 \right) \psi, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^3.$$

To simplify the study at last for the mathematical point of view, one perfoms a nondimensionalization to get rid of a huge number of physical parameters and a reduction of the space-dimension, using shaped confinements. One usually obtains a semiclassical systems of nonlinear Schrödinger equations (see for example [2]), where the semiclassical parameter ε contains several physical objects (Planck's constant, mass of particles, etc). This method is often applied to simplify studies of physical systems described by PDEs.

The new equations are nonlinear versions of the ones that describe molecular dynamics within Born-Oppenheimer approximation ([14, 5, 26, 33, 28, 27, 32]). Molecular dynamics has been widely studied (non-exhaustive list) by Colin de Verdière, Fermanian Kammerer, Gérard, Hagedorn, Joye... Those results allow us to get information on some nonlinear systems, sometimes seen as a perturbative case of the linear ones. Thus in some settings, the nonlinear systems inherit the linear difficulties (tunneling effects or transition phenomena due to eigenvalue crossings) but also have technical obstructions to deal with, coming from the nonlinear part. An important question is the role of the nonlinearity on known linear phenomena and interaction between both difficulties.

In the study of semiclassical nonlinear systems, one considers $d, N \geq 1$ and V, a $N \times N$ hermitian matrix V and the following equation when $\varepsilon \to 0$

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - V(x)\psi^\varepsilon = \kappa\varepsilon^\beta|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d, \qquad (3)$$

for some value of $\beta > 0$ chosen to exhibit nonlinear effects for some initial data. Let us now consider coherent states, built thanks to one chosen mode (the $j-$th eigenvalue and associated eigenvectors of V) as they are simple and useful objects to manipulate:

$$\psi^\varepsilon(0,x) = \varepsilon^{-d/4}a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)e^{i\xi_0(x-x_0)/\varepsilon}\chi_j(x), \quad a \in \mathcal{S}(\mathbf{R}^d).$$

In the linear case, depending on the potential, the situation can be divided into two kind of physical phenomena: either one can prove a kind of stability, adiabatic decoupling of the solution up to some time (one can write the system as N decoupled equations) or one has transition phenomena.

An interesting problem in linked to the latter phenomenon (tunneling effect): in the linear case, when V depends on ε in some way, if its eigenvalues come close to one another without crossing (we say that there is an avoided crossing), then adiabatic decoupling is false. A typical example of such system is (3) with $\kappa = 0$ and $V^\varepsilon(x_1,\ldots,x_d) = \begin{pmatrix} x_1 & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -x_1 \end{pmatrix}$. To deal with a nonlinear version of this problem, one needs to carefully handle the nonlinearity and its coupling effect. Chosing appropriate parameters κ, β, a result similar to the linear one has been proved in [22], but only for $d = 1$. In fact, this case in mass-subcritical,

and energy estimates are sufficient to deal with the nonlinearity. Besides, an other tool, the Lens transform can be used in that precise framework, preventing us from making generalization of such transition results to higher dimensions. One last obstruction to perform a complete study of - at least- this explicit $1d-$case in the nonlinear context, is the lack of tools to prove superposition principle here. In fact, the proof in the linear case [15] is divided into three steps: consider $N = 2$ and two modes $+$ and $-$. Write $\varphi_\pm^\varepsilon$ the coherent states built with the classical trajectories. Then

1. Study the system far from the crossing region as in adiabatic cases (with the adiabatic approximation). At the initial time, denoted -T, one has $\varphi_+^\varepsilon(-T, x)\chi_+$ and one proves that at some $-t^\varepsilon$, the solution is close to $\varphi_+^\varepsilon(-t^\varepsilon, x)\chi_+ + o(1)$.

2. The crossing occurs at time $t = 0$. Consider the region $[-t^\varepsilon, t^\varepsilon]$ as a black box and use asymptotical (scattering type) results to deduce what happens at t^ε. One can write an approximation as $\varphi_+^\varepsilon(+t^\varepsilon, x)\chi_+ + \varphi_-^\varepsilon(+t^\varepsilon, x)\chi_- + o(1)$.

3. Consider the region $[t^\varepsilon, T]$ as in Step (1) and use the linear superposition of coherent states.

In [22], the third step is not proved since nonlinear superposition principles are not proved in that particular case. However, the problem does not seem to be hopeless, thanks to numeric simulations ([21]).
Among many questions that arise in that field, I focused on one specific problem, that is, potential with some exact crossing.

2.2   Crossing of codimension 1 or "smooth" crossing

The potential I consider has a "codimension 1" crossing : as an example, one can consider $d = 2$, v_0 a smooth function, and $\theta \in C_0^\infty(\mathbf{R}^2)$, the following matrices have a codimension one crossing:

$$V_1(x) = v_0(x)\mathrm{Id} - \frac{x_1}{1 + x_1^2 + x_2^2}\begin{pmatrix} \cos(\theta(x)) & \sin(\theta(x)) \\ \sin(\theta(x)) & -\cos(\theta(x)) \end{pmatrix}$$

and

$$V_2(x) = v_0(x)\mathrm{Id} - \frac{x_1}{1 + x_1^2 + x_2^2}\begin{pmatrix} -\cos^2(\theta(x)) & \cos(\theta(x))\sin(\theta(x)) \\ \cos(\theta(x))\sin(\theta(x)) & -\sin^2(\theta(x)) \end{pmatrix}.$$

Then, renaming the energy levels, such that we have smooth quantities, the wave function remains on the smooth energy level at leading order - but higher order terms need a correction. This is similar to the linear case.

A paper is currently being finished to present these results ([20])

3   Conferences and Talks

During the period, I attended some scientific events and was able to add contributions to some of them (this is not a complete list):

Date	Event	Location
June 2016	Conference on Spectral theory,	Cergy, France.
June 2016	Annual meeting of "PDEs workgroup",	Roscoff, France.
March 2016	Young PDE-ists meeting 2016,	Bordeaux, France (**speaker**).
February 2016	8th annual meeting of "DynQua workgroup",	Grenoble, France.

Most of the time, the RCC2 allowed me to attend these events.

4   Miscelleanous

I would like to add some points to this report

1. First, I would like to mention that thanks to the RCC1 provided by the project, I was able to get a new computer since my old one didn't work well.

2. Thanks to various visiting journeys in french universities, financed by the RCC2, I was able to "diffuse" my results and work on the projects.

3. During the nine months of my position I was able to achieve some points to integrate myself: I got the C1 level (CLI of Pisa) for the knowledge of the italian language, which made easier the communications with mathematicians at the University of Pisa and will surely be crucial to go on on these collaborations.

5   Conclusion

Thanks to the fellowships and all facilities provided by INdAM and University of Pisa, I was able to work on my projects in optimal conditions. As a consequence, I obtained a permanent position in France, beginning in september 2016, at the University of Bourgogne Franche Comté, Besançon, as a "Maitre de conférences" ("assistant professor"), and currently pursue my work, initiated at the university of Pisa : I invited some colleagues of Pisa and went back there several times to work, using the network I was able to build thanks to the INdAM-Confund fellowship.

07 October 2017,   Dr. Lysianne Hari

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