Report of the second year

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Graduate Research Topic: On 3 Dimensional Topological Quantum Field Theories: Chern-Simons theory and Dijkgraaf-Witten theory

Research activity
During the year 2019/2020, I worked on the project related Low-dimensional Topology and Quantum Field Theory (QFT) with my supervisor, Pavel Putrov (ICTP).
In 80-90’s, an idea is proposed by Sir. Micheal Atiyah, and E.Witten used QFT to construct Jones polynomial invariant in Knots theory. Topological Quantum Field Theories is a functor from topological spaces to Hilbert spaces. In the dimensional 3, TQFT is studied a structure of Modular Tensor Categories. Example, there are 2 models: Chern-Simons theory (construct by Lie algebra/continous Lie group) and Dijkgraaf-Witten theory (Finite group).
In general Chern-Simons (equivalent, Witten-Reshetikhin-Turaev) Topological Quantum Field Theory is quite different from Dijkgraaf-Witten theory. But for some particular Lie algebras they should be equivalent, this is what the "physics" path-integral analysis predicts.
Our work is to make this a mathematically precise statement and prove the predict the result from path integral in QFTs. Moreover, in the mathematical side, we try to understand structure of quantum groups, representation of quantum group and, in the physical side, we would like to understand the mathematical aspect of quantum field theory. For more detail and references, we refer a file attached.

Courses attended:
(∗) 4-manifolds, Lecture: Rafael Torres, 40 hours, SISSA
(∗) Noncommutative Geometry I, Lecturer: Paolo Antonini, 30 hours, SISSA

Seminar attended:
(∗) regular Ph.D seminar Baby Geometry and regular Geometry seminar of Department of Mathematics, University of Pisa, 2019/2020.
(∗) Basics notion seminar in ICTP, Trieste, 2019/2020.
(∗) regular Seminar on Analytic Geometry and Seminar on Geometry and Topol-

1 Abdus Salam International Center for Theoretical Physics, Trieste, Italy
2 Scuola Internazionale Superiore di Studi Avanzati/International School for Advanced Studies

Seminar talks given

Schools, Workshops, Conferences attended:
(∗) Research School Winterbraids X, Pisa, Italy, 17 February 2020 - 21 February 2020

Visiting period to other universities/Institutes:

Papers published

Papers submitted

Preprints

30/11/2020

Nguyen Anh Hung
Project on 3 dimensional Topological Quantum Field Theory: Chern-Simons theory and Dijkgraaf-Witten theory

There is my summary/report plan of the project in 3d TQFTs. We try to understand the relations between certain Dijkgraaf-Witten and Chern-Simons topological quantum field theories (TQFTs). The path-integral formulation of the theories is quite different, Dijkgraaf-Witten theories have discrete gauge fields while Chern-Simons theories have continuum gauge fields. The mathematical description of these TQFTs via underlying modular tensor categories are also in general different. Nevertheless, my supervisor Pavel Putrov suggested that it is possible to show in certain cases the resulting topological quantum field theories are actually equivalent. These TQFTs have important applications in 3-dimensional topology and condensed matter theory.

1 What is a TQFTs?

In gauge theory and mathematical physics, a topological quantum field theory (or topological field theory or TQFT) is a quantum field theory which focuses on topological invariants. That means the correlation functions do not depend on the space-time in the sense path integral.

1.1 Path integral as motivation

In physics, the formula of path integral is:

\[ Z(M) = \int D\Phi e^{-S[\Phi]} \]  

where:

- \( \Phi \) is a smooth map between smooth manifolds \( M, X \). Once can choose a scalar field \( \Phi, X = \mathbb{R} \).
- \( S[\Phi] \) is called the action functional, the function is on the space of the map from \( M \) to \( X \) (moduli spaces/the space of field configurations). For example of massless free scalar field:

\[ L = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \quad S[\Phi] = \int_M L(\Phi, \partial_\mu \Phi)(x) \sqrt{\det(g)} d^nx, \]

- \( \int D\Phi \) is the integral over the mapping space of field from \( M \) to \( X \).

In general, \( \int D\Phi \) is ill-defined measure, non-rigorous mathematical definition, but it is good tool come from physics to predict new invariants in topology.

Let \( \Theta_1, \Theta_2, \ldots, \Theta_n \) be some observables, that is, complex functions on the space of fields. These could be \( \Theta_i \in \left\{ \Phi(x_0); \partial \Phi(x_0); \Phi^2(x_0); \ldots \right\} \) probe the field at single point \( x_0 \).

The correlation function of the observables \( \Theta_1, \Theta_2, \ldots, \Theta_n \) is

\[ < \Theta_1 \Theta_2 \ldots \Theta_n >_g = \int D\Phi \Theta_1 \Theta_2 \ldots \Theta_n e^{-S[\Phi]}, \]

where \( g \) indicates the metric of spacetime \( M \). If this correlation functions is not dependent on the metric \( g \) of \( M \), we have a topological quantum field theory. The most famous example of a TQFT is Chern-Simons theory \([Wi89]\).

The ingredients:

- a 3-dimensional compact oriented manifold \( M \),
- a compact Lie group \( G \), which it will take to be \( SU(N) \),
• a principal $G$–bundle $P \to M$ (which, as $G = SU(N)$, can always be trivialised $^1$),

• a connection 1-form $A \in \Omega(M, g)$, for $g$ the Lie algebra of $G$.

Here $A$ plays the role of field as $\Phi$ above, and the functional action is

$$S = \frac{k}{4\pi} \int_M \text{tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \quad (3)$$

The observables are Wilson loops$^2$, a gauge invariant operator. More concretely, given an irreducible representation $R$ of $G$ and a loop $K$ in $M$, one may define the Wilson loop $W_R(K)$ by

$$W_R(K) = \text{Tr}_R P \exp \left( i \oint_K A \right) \quad (4)$$

where $P \exp$ is the path-ordered exponential.

Consider a link $L$ in $M$, which is a collection of $k$ disjoint loops. One may form a normalized correlation function by dividing this observable by the partition function $Z(M)$, which is just the 0–point correlation function.

In the special case in which $M$ is the $3$–sphere, Witten has shown that these normalized correlation functions are proportional to known knot polynomials$^3$. Some properties we expect from TQFTs: Functoriality, Gluing, Normalization, we will see it later in the mathematical formulate of TQFTs.

### 1.2 TQFTs as functors

All of $(d + 1)$–dimensional space-time (bordism) can consider as symmetric monoidal tensor categories, we refer the book [EGNO15] for more detail. $(d + 1)$–dimensional TQFT studies structure of bordism categories$^4$.

**Definition** An $(d + 1)$–dimensional oriented TQFT is a symmetric monoidal functor:

$$Z : \text{Bord}_d \to \text{Vect}_k$$

(a) To any $d$-manifolds $N$ without boundary is assigned a finite-dimensional vector space $\mathcal{T}(N)$.

(b) To any $(d + 1)$–manifold $M$ (possibly with boundary) is assigned a vector $\mathcal{T}(M)$ in the vector space $\mathcal{T}(\partial M)$.

(c) To any homeomorphism of $d$–manifolds $f : N \to N'$ is assigned an isomorphism of vector spaces $f_* : \mathcal{T}(N) \to \mathcal{T}(N')$.

(d) Functorial isomorphisms

$$\mathcal{T}(\bar{N}) \xrightarrow{\cong} \mathcal{T}(N)^*, \mathcal{T}(\emptyset) \xrightarrow{\cong} k, \mathcal{T}(N_1 \sqcup N_2) \xrightarrow{\cong} \mathcal{T}(N_1) \otimes \mathcal{T}(N_2)$$

where $\bar{N}$ is the manifold $N$ with the opposite orientation, which are compatible in an obvious sense with each other and with commutative, associative and unit morphism.

These data are required to satisfy the following axioms:

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$^1$Lemma: If $M$ is a compact orientable manifold of dimension $\leq 3$, and the homotopy groups $\pi_0(G)$ and $\pi_1(G)$ are trivial, then $P$ admits a global section, i.e $P$ is trivial $G$–bundle

$^2$Holonomy operators

$^3$a knot polynomial is a knot invariant in the form of a polynomial whose coefficients encode some of the properties of a given knot, example: Jones polynomial, Alexander polynomial,....

$^4$the category $\text{Bord}_d$ has

(i)as objects closed $d$–dimensional smooth manifolds

(ii)and the morphisms are compact $(d + 1)$–dimensional smooth manifolds with boundary, modulo diffeomorphism “rel boundaries” (i.e. those that restrict to the identity on the boundary)

The composition of morphism is given by gluing of manifolds along their boundary
(i) **Functoriality.** If \( f : M \xrightarrow{\sim} M' \) is a homeomorphism of \((d + 1)-\)manifolds then \((f |_{\partial M})_* (\mathcal{T}(M)) = \mathcal{T}(M')\).

(ii) **Gluing axiom.** Let \( M \) be a \((d + 1)-\)manifolds, \( \partial M = N_1 \sqcup N_2 \sqcup N_3 \), and \( f : N_1 \to \bar{N}_2 \) be a homeomorphism. Let \( M' = M / f \) be the \((d + 1)-\)manifold obtained from \( M \) by identifying \( N_1 \) with \( \bar{N}_2 \) using \( f \), i.e., by gluing \( N_1 \) to \( N_2 \). Then \( \mathcal{T}(M') \) is equal to the image of \( \mathcal{T}(M) \) via the map \( \mathcal{T}(N_1) \otimes \mathcal{T}(N_2) \otimes \mathcal{T}(N_3) \to \mathcal{T}(N_2) \otimes \mathcal{T}(N_3) \to \mathcal{T}(N_3) \).

(iii) **Normalization axiom.** Let \( I \) be an interval and \( N \) be a \( d-\)manifold. Then \( \partial(I \times N) = N \sqcup \bar{N} \) and we require that \( \mathcal{T}(I \times N) \) equals the image of \( id_{\mathcal{T}(N)} \) in \( \mathcal{T}(N) \otimes \mathcal{T}(N) \simeq \mathcal{T}(N)^\ast \otimes \mathcal{T}(N) \).

(iv) \( \mathcal{T}(S^2) = k, \mathcal{T}(B^{d+1}) = 1 \in k \), where \( B^{d+1} \) is the unit ball in \( \mathbb{R}^{d+1} \), and \( S^d = \partial B^{d+1} \).

**Theorem.** \((*) \)(0+1) dimensional TQFTs is just a finite dimensional Hilbert space.

\((*) \) (1 + 1) dimensional TQFTs are in one-to-one correspondence with finite-dimensional Frobenius algebras \(^5\) i.e., commutative associative algebras \( A \) with unit and with a linear map \( tr : A \to k \) such that the bilinear form \( tr(ab) \) is non-degenerate.

Bigg Theorem any 3 dimensional TQFT into abelian categories\(^6\) over \( \mathbb{C}(\text{AbCat}_\mathbb{C}) \) must be a modular tensor category.

Higher dimensional TQFTs is hard to study, moreover, It is still not clear the picture of classification of \((d+1)\) dimensional TQFTs for \( d > 3 \).

2 On 3 dimensional TQFTs

Here we give two deep theories are connected Knot theory and 3-manifolds.

**Theorem**(Lickorish-Wallace)

Every connected oriented closed 3-manifold \( 4N \) arises by performing an integral Dehn surgery\(^7\) along

\(^5\)Read [Ko03] for more detail of \((1+1)\)TQFTs and Frobenius algrba.

Definition A Frobenius algebra in a monoidal category is a quintuple \( (A, \delta, \epsilon, \mu, \eta) \) such that:

(i) \((A, \mu, \eta)\) is a monoid with multiplication \( \mu : A \otimes A \to A \) and unit \( \eta : I \to A \),

(ii) \((A, \delta, \epsilon)\) is a comonoid with comultiplication \( \delta : A \to A \otimes A \) and counit \( \epsilon : A \to I \), and

(iii) the Frobenius laws hold: \((1 \otimes \mu) o (\delta \otimes 1) = \delta o \mu = (\mu \otimes 1) o (1 \otimes \delta),\)

\(^6\)refer category theory in (2.1) [EGNO15]

\(^7\)Dehn surgery contains two process: Dehn drilling and Dehn filling.

(i) **Dehn drilling.** Form the complement in the 3-sphere of a tubular neighborhood of the embedded link \( L \) in \( S^3 \), of the form \( \text{Lint}(D^2) \). The result is a 3-manifold with boundary \( M \), whose boundary \( \partial M \simeq L \times S^1 \) can be viewed as
a framed link\(^8\) \(L\) in \(S^3\) (i.e., surgery along a framed link). Denote this manifold is \(M_L\).

**Theorem** (Kirby Calculus)

\(M_L \simeq M_{L'}\) iff \(L'\) can be obtained from \(L\) by the sequence of Kirby-Fenn-Rourke moves shown in the picture below.

One of algebraic way to studying invariant of Knot Theory is closed related to representation theory quantum groups (generally, tensor categories, braided tensor categories, modular tensor categories, etc...)\(^{[Ye01]}\). Here we need to introduce the concepts Hopf algebra and monoidal categories (restrict to category of module over a algebra with extra structure).

### 2.1 Knot invariants from categories

**Definition.** A \(k\)-bialgebra \((A,m,\eta,\Delta,\epsilon)\) with multiplication \(m\), comultiplication \(\Delta\), unit \(\eta : k \to A\) and counit \(\epsilon : A \to k\) is called a Hopf algebra if there exists a \(k\)-linear function \(S : A \to A\), called antipode such that

\[
m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon
\]

**Example:**

(i) \(k[G]\) group algebra of finite group \(G\).

\[
m(x,y) = xy, \quad \Delta(x) = x \otimes x, \quad \eta(1) = e, \quad \epsilon(x) = 1, \quad S(x) = x^{-1}.
\]

(ii) Universal enveloping Lie algebra \(U(g)\).

\[
m(x,y) = x \otimes y, \quad \Delta(x) = 1 \otimes x + x \otimes 1, \quad \eta(x) = 0, \quad \epsilon(1) = 1, \quad S(x) = -x.
\]

**A category theory.** Roughly speaking, monoidal category is a category have monoidal structure on the objects, denote this operation \(\otimes\) namely, tensor and unit object satisfy a certain condition as unit element in the monoid(group).

Example: (*)Set category is tensor category with tensor be a Cartesian product,

(*) Category of finite dimensional vector space.

**Strict Braid Category** A commutativity constraint \(c\) on a monoidal category \(\mathcal{C}\) is a choice of isomorphism \(c_{A,B} : A \otimes B \to B \otimes A\) for each pair of objects \(A\) and \(B\) which form a "natural family."

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\(^8\)Intuitively, a framed knot or link may be thought of as a knot or link formed from a length of ribbon rather than a length of string.
A braiding is a commutativity constraint in $C$ satisfies the relations:

\[ c_{U \otimes V, W} = (c_{U, W} \otimes id_V) \circ (id_U \otimes c_{V, W}) \]
\[ c_{U, V \otimes W} = (id_V \otimes c_{U, W}) \circ (c_{V, W} \otimes id_U) \]

A braided monoidal category is a monoidal category with a braiding. Let fix $A$ be a algebra over $K$ and $\mathcal{Rt}_{A}(A)$ category of finite dimensional.

A **braided algebra** is an algebra such that $\mathcal{Rt}_{A}(A)$ is a braided monoidal tensor category. Braided algebra is characterised by universal $R$–matrix in $A \otimes A$.

**Ribbon Hopf algebra** is a braided and Hopf algebra with an invertible central element $\nu$ more commonly known as the ribbon element, such that the following conditions hold:

\[ \nu^2 = uS(u), \quad (\nu) = \nu, \quad \epsilon(\nu) = 1, \quad \text{and} \quad \Delta(\nu) = (R_{21}R_{12})^{-1}(\nu \otimes \nu) \]
where $u = m(S \otimes id)(R_{21})$.

Once can define Ribbon category as a category of finite-dimensional representation of Ribbon algebra. In [RT90], [RT91], Reshetikhin and Turaev constructed new invariant of knot/link/tangle from Ribbon Hopf algebra as well from ribbon category.

A **ribbon category** is a rigid braided monoidal category equipped with a twist be compactible

\[ (\theta_C \otimes id_C^*)b_C = (id_C \otimes \theta_C^*)b_C, \]

- **Rigid.** each object $C$ there is another object (called the left dual), $C^*$, with maps $b_C : 1 \to C \otimes C^*$ and $d_C : C^* \otimes C \to 1$ such that the compositions

\[ C^* \cong C^* \otimes 1 \xrightarrow{id_C^* \otimes b_C} C^* \otimes (C \otimes C^*) \cong (C^* \otimes C) \otimes C^* \xrightarrow{d_C \otimes id_{C^*}} 1 \otimes C^* \cong C^* \]

equals the identity of $C^*$, and similarly with $C$.

- A **twist** on rigid braided monoidal category is a set of isomorphisms $\theta_X : X \to X$ for which

\[ \theta_{X \otimes Y} = c_{Y,X}c_{X,Y}\theta_X \otimes \theta_Y \]

**Trace and quantum dimensional** in ribbon category $C$

Denote by $K = K_C = End(1)$ a commutative semigroup with multiplication induced by composition of morphisms and the unit element $id_1$. For $V \in C$, $f \in End(V)$, define

\[ tr(f) = d_V c_{V,V^*}(\theta_V \circ f \otimes id_{V^*})b_V : 1 \to 1 \]
\[ dim(V) = tr(id_V) = d_V c_{V,V^*}(\theta_V \otimes id_{V^*})b_V : 1 \to 1 \]

**Additive category** Let $K$ is commutative ring. A monoidal category is additive with ground ring $K$ if Hom-set in $C$ is $K$–module, composition and tensor product of morphisms are $K$–linear, and the formula $k \to k.id_1$ defines an isomorphism $K$–module from $K$ to $End(1)$.

**Simple Object** An object $V$ in an additive monoidal category is simple if the formula $k \to k.id_V$ defines an isomorphism $K$–module of $K$ to $End(V)$.

We can think on category representation theory of group algebra $k[G]$, a simple object is irreducible representation of $G$ (as well simple $k[G]$module).
2.2 Modular Tensor Categories (MTC)

A **modular tensor category** is a pair \((C, \{V_i\}_{i \in V})\) consisting of an additive ribbon monoidal category \(C\) with ground ring \(K\) and a finite family \(\{V_i\}_{i \in V}\) of simple objects of \(C\) satisfying the following four axioms.

(i) (Normalization axiom.) There exist \(0 \in I\) such that \(V_0 = 1\).

(ii) (Duality axiom.) For any \(i \in I\) there exists \(i^*\) such that the object \(V_{i^*}\) is isomorphic to \((V_i)^*\).

(iii) (Axiom of domination.) All objects of \(C\) are dominated \(^9\) by \(\{V_i\}_{i \in I}\).

(iv) (Non-degeneracy axiom) For \(i, j \in I\), set \(S_{i, j} = tr(c_{V_j, V_i} \circ c_{V_i, V_j}) \in \text{End}(1) = K\) The square matrix \(S = (S_{i, j})_{i, j \in I}\) is invertible over \(K\).

**Definition/ Denote.** Suppose \(C\) is a modular tensor category, fix \(K = C\). Let

- \(\theta_i := \theta_{V_i} \in \text{End}(V_i) = K\) where \(\theta\) is the balancing natural isomorphism in \(C\);
- \(d_i := \text{dim}(V_i)\) quantum dimensional of object \(V_i\);
- \(p^\pm = \sum_{i \in I} \theta_i^{-1} d_i^2, \quad D^2 = p^+ p^- = \sum_{i \in I} d_i^2\).

**Definition and Theorem** The Reshetkhin-Turaev invariant of a 3−manifold \(M_L\) is a topological invariant:

\[ T(M) = T_{(C, D)}(M) D^{N-1} \left( \frac{p^+}{p^-} \right)^{\sigma(L)} RT_V(L) \]

where

- \(N\) is the number of components in the link;
- \(\sigma(L)\) is the writhe number of \(L\) (a topological invariant of the link \(L\));
- \(RT(L)\) is the framed link invariant defined as:

\[ RT_V(L) = \sum_{\gamma \in \text{col}(L)} \left( \Pi_{1 \leq n \leq N} d_{q(n)} \right) F(L, \gamma) \in K \]

where \(F\) is a unique functor from \(C\)−colored tangles to strict ribbon tensor category \(C\), \(F(L, \gamma)\) means colored link \(L\) by colored\(^{10}\) \(\gamma\).

The invariant \(T_{(C, D)}\) admits an extension to 3−bordisms (or 3 dimensional TQFTs). In the next sections, we give 2 theories to construct the modular tensor categories of representation of ribbon Hopf algebra.

2.3 Chern-Simons Theory

The idea of E.Witten\(^{[Wi89]}\) defined the topological invariant of 3−manifold associated to non-abelian Lie group (on the physical level of rigor). Perhaps, the parallel mathematical Witten’s theory is Reshetikhin-Turaev constructions associated to semi-simple Lie algebra, namely, quantum group\(^{[RT91]}\).

**Example.** Quantum group \(U_q(\mathfrak{sl}_2)\) as non-(co)commutative Hopf algebra.

Define a algebra generated by \(E, F, K\) and \(K^{-1}\), such that

\[ KK^{-1} = K^{-1} K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F; \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}; \]

\(^9\)in the sense of vector space, dominated means finite generated by object \(\{V_i\}_{i \in V}\)

\(^{10}\)Here colored is a label the component of link by simple object \(\{V_i\}_{i \in I}\)
\( \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K; \)

\[ \eta(E) = \eta(F) = 0, \quad \eta(K) = 1; \]

\[ S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}. \]

Remark: \( \lim_{q \to 1} U_q(\mathfrak{sl}_2) = U(\mathfrak{sl}_2) \) a co-commutative Hopf algebra.

Theorem \( U_q(\mathfrak{sl}_2) \) is a ribbon Hopf algebra.

Let \( C(\mathfrak{sl}_2, \chi) \) be the category of finite dimensional representations of \( U_q(\mathfrak{sl}_2) \) at root of unity over field \( \mathbb{C} \), where \( \chi \in \mathbb{Z}_+ \).

Theorem \( C(\mathfrak{sl}_2, \chi) \) is a ribbon category over \( \mathbb{C} \).

One can try to modify \( C(\mathfrak{sl}_2, \chi) \) to obtain a new modular tensor category by Andersen and Paradowski’s construction, see more detail in \([AP],[BK00] \). Generally, this process can be done for all simple Lie algebra.

### 2.4 Dijkgraaf-Witten theory

In [DW90], Dijkgraaf and Witten discussed the functional integral formulation of finite gauge group Chern-Simons theory by defining a particular TQFT. The path integral is rigorous because the integral is sum over a finite set. Altschuler and Coste [AC92] develop Dijkgraaf-Witten theory from the Reshetikhin-Turaev perspective of link diagrams and certain (quasi-) Hopf algebras. They conjectured the equivalence of their construction with the Dijkgraaf-Witten theory. The equivalence was proven for closed manifolds in [KSW05].

**Dijkgraaf-Witten Invariant**

For a closed oriented 3–manifold \( M \), the Dijkgraaf-Witten invariant [DW90] is given by the state sum,

\[ Z_\theta(M) = \frac{1}{|G|} \sum_{\gamma \in \text{Hom}(\pi_1(M), G)} \langle \gamma^* [\theta]; [M] \rangle. \]

Here \([\theta]\) is the cohomology class of \( H^3(BG, U(1)) \), and \([M]\) is the fundamental class of \( M \). \( \gamma^* \) is the map \( H^3(BG, U(1)) \) to \( H^3(M, U(1)) \) induced by the classifying map \( M \to BG \) corresponding to a representation \( \gamma: \pi_1(M) \to G \). In [WA92], Wakui gave a formulation of the Dijkgraaf-Witten invariant using triangulations. Further he showed that the formulation can be extended for 3–manifolds with boundaries, and the construction gives an example of the topological quantum field theory.

**Quantum double of a finite group**

Let fix \( G \) be a finite group. Recall the group algebra \( k[G] \) over field \( k \) is a co-commutative Hopf algebra with a \( k \)–basis \( \{ x \}_{x \in G} \) and

- multiplication \( x \otimes y \to xy \), unit \( e \) (the unit element of \( G \)),
- comultiplication \( \Delta(x) = x \otimes x \), counit \( \epsilon(x) = 1 \),
- antipode \( S(x) = x^{-1} \).
By Maschke’s theorem, the category $\mathcal{Rep}_k[G]$ of finite dimensional representation is semi-simple ($k[G]$—module is the same as a representation of $G$).

The dual of Hopf algebra $k[G]$ is isomorphism to the function algebra $F[G]$ with $k$—basics $\{\delta_g\}_{g \in G}$.

Using Drinfeld’s quantum double construction [DR87], on can obtain $D(G) = F[G] \otimes_k k[G]$ - a quasi-triangular Hopf algebra with:

- multiplication $(\delta_g \otimes x)(\delta_h \otimes y) = \epsilon_{gx,h} \delta_{gxy}$, $x, y, g, h \in G$;
- co-multiplication $\delta(\delta_g \otimes x) = \sum_{g_1g_2=g} (\delta_{g_1} \otimes x)(\delta_{g_2} \otimes x)$,
- unit $1 = \sum_{g \in G} \delta_g \otimes e$, counit $\epsilon(\delta_g \otimes x) = \epsilon_{g,e}$
- antipode $S(\delta_g \otimes x) = \delta_{x^{-1}g^{-1}x} \otimes x^{-1}$.

Let $\mathcal{Rep}_k D(G)$ be the category of finite-dimensional representations of $D(G)$ as a $k$-algebra.

**Theorem** $\mathcal{Rep}_k D(G)$ is a modular tensor category.

### 3 Project

Namely, the question is to show equivalence of the two TQFTs:

1. Dijkgraaf-Witten theory for $\Gamma = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ group (i.e. described by twisted quantum double in terms of MTC).

Given a finite abelian group $G = Z_N^3$, $H^3(BG, U(1)) = Z_N^3 \times Z_N^3 \times Z_N$ and taking twist $\alpha = (0,0,p) \in H^3(BG, U(1))$.

2. Chern-Simons theory for the Lie algebra $\mathfrak{g}$ with 6 generators $a_i, b_i$, $i = 1, 2, 3$, and commutation relations.

   $[a_i, a_j] = \sum_k \epsilon_{i,j,k} b_k$ and Ad-invariant inner product

   $\langle a_i, b_j \rangle = \delta_{i,j}$ and $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$

To be more concrete, a simplest naive guess would be that for the Dijkgraaf-Witten theory with group $Z_N^3$ and a co-cycle specified by $p \in Z_N \subset H^3(Z_N^3, U(1))$ (where $Z_N$ is the subgroup which mixes all three $Z_N$).

The corresponding quantum group is an algebra with generators $F_i, K_i, K_i^{-1}$, $i, j, k = 1, 2, 3$ and relations:

- $[F_i, F_j] = \epsilon_{ijk}(K_j^p - K_j^{-p})/(q - q^1)$
- $K_i K_i^{-1} = 1$
- $K_i F_j = F_j K_i$
- $K_i K_j = K_j K_i$

with $q = exp(\pi i/k)$.

When $q > 1$ this indeed becomes the universal enveloping of the Lie algebra

- $[F_i, F_j] = p \epsilon_{ijk} E_k$
- $[F_i, E_j] = 0$
- $[E_i, E_j] = 0$ assuming $K_i = q^{E_i}$.

The quantum version depends only on $p \mod k$. Then the first step would be to see what are possible finite dimensional representations of the algebra above (one can answer this question for some small values of $k$ and $p$ first) and then check if they can be made in one-to-one correspondence with the simple objects of the MTC in DW.
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