PhD Report 2nd year

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1 Research Field

The overall goal of the working group at the University of Rome "Tor Vergata", that I am part of is to study the connection between the theory of Operator Algebras and the mathematical foundations of Quantum Field Theory (QFT).
The axiomatization, which most of our work is based on, is due to Haag and Kastler. The central object is a local net of von Neumann algebras assigning to each space-time region $\mathcal{O}$ a von Neumann algebra $\mathcal{A}(\mathcal{O})$

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \tag{1}$$

that behaves well under space-time symmetry. The elements in $\mathcal{A}(\mathcal{O})$ are interpreted as observables that can be measured by an experiment in $\mathcal{O}$.

One of the successes of Haag's and Kastler's axiomatic approach - known as Algebraic QFT (AQFT) - is the deeper conceptual understanding of superselection sectors. Arguably the most important works about superselection theory in AQFT were contributed by Doplicher, Haag and Roberts (DHR). Unfortunately, the assumptions in the DHR theory exclude long-range forces like electromagnetism.
It is an active area of research to generalize the results of DHR. In particular, the theory of Quantum Electrodynamics (QED) is of special interest.
I collaborated with W. Dybalski, TU Munich, on superselection sectors linked with gauge invariance in external current QED. The collaboration resulted in [1], which was published on 22 November 2019 in the Journal of High Energy Physics.
In this work, we identified different superselection sectors in a model of interacting QED. These superselection sectors stem from different choices of fixing the U(1)-gauge freedom of electrodynamics. We linked our findings to the Aharanov-Bohm effect. We hope that these results can be used as a starting point for the development of superselection theory of local gauge invariance in the spirit of DHR.

In the last years, the interplay between Quantum Information Theory and QFT, that is originated in the investigation of black hole thermodynamics, has been subject of deep analyses.
One of the basic quantities in information theory is entropy. The quantum mechanical version of Shannon entropy is called von Neumann entropy. However, in AQFT, the local algebras are usually type III_1-factors. Thus, no trace or density matrix exists. The von Neumann entropy, hence, is undefined for QFT. However, using Tomita-Takesaki theory, one can define a notion of relative entropy between normal faithful states, that extends the well-known Umegaki relative entropy for type I-factors.
The notion of relative entropy for QFTs can be used to define and investigate certain energy conditions in general relativity. Energy conditions are restrictions to prevent unphysical solutions of the Einstein equations. An example of a relatively weak condition is the null energy condition (NEC). However, the NEC suffices to prove many important results such as the celebrated Penrose-Hawking singularity theorem.
Anyway, QFTs violate all local energy conditions including the NEC. Thus, finding generalizations of the NEC in presence of Quantum Fields is an important issue that can help to generalize results in general relativity and provide deeper inside in the nature of Quantum Gravity. One of the candidates to generalize the NEC is the Quantum Null Energy Condition (QNEC).
One energy condition that has no known counterexamples in consistent QFTs is the Averaged Null Energy Condition. It is the NEC integrated over an entire null direction. Faulkner et al [2] were able to recover the QNEC from the ANEC in certain situations. Their analysis bases on a result due to Casini et al [3] about the Tomita-Takesaki modular operator for distorted wedges.
The expression contains a singular integral over the stress-energy tensor. From a mathematical point of view, it is not clear that this expression is well-defined. In our analysis, we proved that the Tomita-Takesaki operator has a well-defined form for null-cuts for the scalar field in $D+1$ dimensions:

$$\log(\Delta_{H(N_C)}) = \int_{\mathbb{R}^{D-1}}^{\oplus} \log(\Delta_{U(1)}) + 2\pi C(\mathbf{x}_\perp) P_{x_+}(\mathbf{x}_\perp) d^{D-1}\mathbf{x}_\perp. \tag{2}$$

In particular, we are working with restriction of the free scalar field in $D+1$ dimension to the null plane $x_0 - x_1 = 0$ in the Minkowski space. Null-cuts are regions in this null plane defined by continuous functions $C(\mathbf{x}_\perp)$ on the entangling surface. A null-cut is the region defined by $x_0 + x_1 > C(\mathbf{x}_\perp)$. Our analysis start by disintegrating the representation of the subgroup of translations along $x_0 + x_1$ and the boost along x_1 with respect to $\mathbf{x}_\perp$. The restriction on each fibre is equivalent to the well-studied $U(1)$-current, whose Tomita-Takesaki modular operator is known.
The Bisognano-Wichmann theorem implies that the modular operator for the standard subspace associated to the 0-null cut is the generator of boost in x_1-direction. The previously mentioned disintegration encodes that these boosts act as dilations on each fibre. Hence, the modular operator for the 0-null-cut is the direct integral over modular operators of the $U(1)$-current.
For the deduction of (2), we utilized half-sided modular inclusions. It is a powerful tool -

due to [4] - to analyse the modular operators of standard subspace (or von Neumann algebra), that satisfy a certain inclusion property.
To complete the project, we are missing one technical result. It is a kind of duality argument for the standard subspaces associated to null-cuts: $H(N_C) = H(N_C'')$.
We expect to publish our results soon.

However, to verify the QNEC for certain types of regions it is essential to compute the associated Tomita-Takesaki modular operator. The model-independent results of Bisognano and Wichmann connects the modular group of wedge regions W with the Lorentz boosts that leave W invariant. For different types of regions the understanding of the modular group and hence the modular operator is more difficult as it does not have a geometrical meaning. By the Reeh-Schlieder property the vacuum vector is standard for every local algebra $\mathcal{A}(\mathcal{O})$ if $\mathcal{O}$ and its complement $\mathcal{O}^c$ have non-empty interiors. Therefore, the associated modular theory exists and we can define the modular operator accordingly. One of the basic regions in QFT models are double cones, the causal envelopes of a ball. In free massless QFT, the geometric description of the double cone vacuum modular group was derived in [5]. In [6] the authors explicitly state the modular Hamiltonian $\log(\Delta_{H(\mathcal{O})})$ for the double cone $\mathcal{O}$ in this situation in terms of its Cauchy data:

$$\log(\Delta_{H(\mathcal{O})}) = -2\pi i \begin{pmatrix} 0 & M \\ L & 0 \end{pmatrix}, \tag{3}$$

where M is a multiplication operators and L a Legendre operator. Moreover, the authors extended this result to the massive case in $3+1$ and $2+1$ dimensions to

$$\log(\Delta_{H(\mathcal{O})}) = -2\pi i \begin{pmatrix} 0 & M \\ L_m & 0 \end{pmatrix}, \tag{4}$$

where L_m is the sum of the "massive" L and a Greens function integral operator. Their analysis based on a certain deformation argument of the massless to the massive case, that fails in $1+1$ dimension. However, it is expected that the modular Hamiltonian of the double cone for the free massive scalar field has the form in $1+1$ dimensions. A first indication can be found in [7], where the authors verify the conjecture on a subspace. We hope to verify this generalization to $1+1$ dimensions overall.

2 Courses and Exams

- **Italian for Foreigners: A2**
  Place: University of Rome "Tor Vergata"
  Period: Oct - Dec 2019, 60 hours

- **Quantum Mechanics**
  Place: University of Rome "La Sapienza"
  Held by: Prof. Doplicher
  Exam: Oral, 30/30, Feb 2020

- **Relative boundary-bulk duality and orbifold subfactors**
  Place: University of Rome "Tor Vergata"
  Held by: Prof. Kawahigashi
  Period: 12 - 21 Feb 2020, 5 hours

- **Operator Algebraic Models in Quantum Field Theory**
  Place: University of Rome "Tor Vergata", online
  Held by: Dr. Ciolli
  Exam: Jun - Jul 2020

3 Research activities

- **Conference**
  Title: Operator algebras and quantum probability
  Place: University of Rome "Tor Vergata"
  Date: 4 - 7 Dec 2019

- **Seminar**
  Title: Seminar on Operator algebras
  Place: University of Rome "Tor Vergata"
  Date: irregular

- **Seminar**
  Title: Young Researchers' Seminar
  Place: University of Rome "Tor Vergata", "La Sapienza" and "Roma Tre"
  Date: irregular

- **Seminar**
  Title: Math physics learning seminar
  Place: University of Connecticut, online
  Date: weekly, 18 Sep 2020 - ongoing

- **Seminar Talk**
  Title: Operator Algebras and Quantum Field Theory
  Seminar title: Math physics learning seminar
  Place: University of Connecticut, online
  Date: 18 Sep 2020

- **Several Online Talks**
  Date: Apr 2020 - ongoing

4 Publications

- [1], published in "Journal of High Energy Physics" in Nov 2019

References

[1] W. Dybalski, B. W. *Asymptotic charges, large gauge transformations and inequivalence of different gauges in external current QED*. J. High Energ. Phys. 2019, 126 (2019)

[2] F. Ceyhan, T. Faulkner *Recovering the QNEC from the ANEC*. `arXiv:1812.04683` [hep-th].

[3] H. Casini, E. Teste, G.Torroba *Modular Hamiltonians on the null plane and the Markov property of the vacuum state*. Journal of Physics A: Mathematical and Theoretical. 2017, 50 (2017)

[4] H.W. Wiesbrock. *Half-sided modular inclusions of von Neumann algebras*. Commun. Math. Phys. 157, 83-92 (1993)

[5] P.D. Hislop, R. Longo *Modular structure of the local algebras associated with the free massless scalar field theory*, Commun. Math. Phys. 84 (1982), 71–85.

[6] R. Longo, G. Morsella *The massive modular Hamiltonian*. `arXiv:2012.00565` [math-ph].

[7] R.Conti, G. Morsella *Asymptotic Morphisms and Superselection Theory in the Scaling Limit II: Analysis of Some Models*, Comm. Math. Phys. 376 (2019), 1767–1801