

February 3, 2021

INdAM-DP-COFUND-2015 FINAL REPORT

Dear Prof. Dr. Giorgio Patrizio,

I am glad to inform you that I presented my scientific research works to the faculty board of the host PhD Programme at *Politecnico di Milano* on October 22, 2020. The board of professors admitted me for the final PhD defense with a passing grade A. Moreover, I would like to inform you that my PhD thesis got two positive referee reports that were submitted by Prof. Paula Cerejeiras from University of Aveiro in Portugal and Prof. Hendrik De Bie from Ghent University in Belgium. The final PhD defense will take place at Politecnico di Milano on February 22, 2021 at 10:30 (Italian time) in front of the committee:

- Prof. Dr. Paula Cerejeiras from University of Aveiro (Portugal)
- Prof. Dr. David Kimsey from University of Newcastle (UK)
- Prof. Dr. Elsa Marchini from Politecnico di Milano (Italy)

This report presents the research activities carried out under the project *INdAM Doctoral Programme in Mathematics and/or Applications Cofunded by Marie Skłodowska-Curie Actions*, grant number: 713485. It includes a summary of the main results of the research project, a complete list of attended PhD courses, the list of original publications related to the project, research stays abroad, the main participations in different scientific events including seminars, congresses, talks given in international conferences and workshops, etc.

1 General information

- **Name of the fellow:** *Kamal Diki*
- **University:** *Politecnico di Milano*
- **Doctoral Programme:** *Mathematical Models and Methods in Engineering*
- **Head of PhD Programme:** *Prof. Dr. Irene Maria Sabadini*
- **Supervisor:** *Prof. Dr. Irene Maria Sabadini*
- **Tutor:** *Prof. Dr. Elsa Maria Marchini*
- **Email address:** *kamal.diki@polimi.it / kamal.diki@gmail.com*

2 Research groups and association memberships

Since 2019, I became member of the following national and international research groups and associations:

- INdAM research group GNSAGA, Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.
- ISAAC, International Society for Analysis, its Applications and Computation.
- MCAA, Marie Curie Alumni Association.

3 Research project progress

Research project title: *Reproducing kernel techniques and polyanalytic function theory in hypercomplex analysis*

Summary of the research work: In this research project, we study different reproducing kernel function spaces and associated integral transforms in the setting of complex, quaternions and Clifford analysis. In particular, we focus on some specific examples such as Segal-Bargmann-Fock spaces, Bergman spaces, Hardy spaces and Gabor spaces. These models are very important in complex analysis, operator theory and have several applications in mathematical physics, time frequency analysis and signal processing, especially in quantum mechanics. As it is well-known, in quantum mechanics physical quantities such as position, momentum and energy are represented by operators acting on some complex Hilbert spaces. In 1961, Bargmann constructed a Hilbert space of entire functions on which the creation and annihilation operators are adjoints of each other's and satisfy the classical commutation rules. This space is known as Fock or Segal-Bargmann space, sometimes called also the bosonic Fock space with n degrees of freedom. Moreover, to any particle moving on the real line is associated a wave function which defines a unit vector of the classical Schrödinger Hilbert space. This unit vector is mapped onto a special holomorphic function making use of a particular exponential kernel. The new resulting complex function is the so-called Segal-Bargmann transform. In the last years, this subject attracted several mathematicians and physicists working in the field of Clifford analysis and related topics. As a consequence, many results and research problems were considered and developed in this direction. In particular, in this project we investigated some new reproducing kernels and associated Hilbert spaces using different tools and techniques from complex and Clifford analysis, motivated by such special integral transforms that are involved in several applications in mathematical physics, like Segal-Bargmann transforms and Gabor or short-time

Fourier transforms. Another main contribution that we achieved during this research project is that we initiated exploring a new research path by extending the theory of slice regular or slice monogenic functions to higher order and considering the so-called slice poly-analytic or slice poly-monogenic function theory on which several questions are open now. These functions can be considered from different points of view. A first approach consists of considering the space of quaternions as union of complex planes and to see these functions as null solutions of the n -th power of the Cauchy-Riemann operator with respect to each complex plane. A second approach is based on the so-called poly-decomposition, which makes a slice poly-analytic function of order n obtained as a sum of n slice regular functions multiplied on the left by some conjugate powers.

A third approach consists in considering slice poly-analytic functions in the kernel of the n -th power of a certain global operator with non-constant coefficients. We note also that a generalized version of the polyCauchy formula and the famous Fueter-Sce-Qian mapping theorem in Clifford analysis were also introduced and proved in this framework. This new construction allowed to relate the different poly function theories in hypercomplex analysis. Furthermore, an important fact that was observed is that this slice polyanalytic function theory contains one of the most important subclasses of the Cauchy-Fueter hyperholomorphic function theory, namely the class of Fueter hyperholomorphic functions of axial type. A very natural and interesting problem that has to be considered now is to develop a natural S -functional calculus associated to this new poly-analytic function theory in hypercomplex analysis. In this research project, under the supervision of Prof. Irene Maria Sabadini, I co-authored 8 original research papers, from which 4 are already published, 2 are accepted to appear and 2 are still submitted under review process. In these papers, we dealt with different research problems touching several topics including: slice hyperholomorphic and monogenic function theories, reproducing kernel theory, quaternionic approximation theory, Fock and Bergman spaces, poly-analytic function theory, Dirac operator in Clifford analysis, quaternionic Segal-Bargmann and Fourier transforms, polyFueter mapping theorem and its applications, Clifford-Appell systems, Short-time Fourier transforms and reproducing kernel Gabor spaces, hypercomplex Hardy spaces and Schur analysis, etc. We give a brief overview on the different results obtained in these research articles:

- The first paper is a joint work with my supervisor Prof. Sabadini and Prof. Gal from university of Oradea, Romania. In this work, the Banach Fock spaces of slice hyperholomorphic functions on the quaternions were introduced, both of the first and of the second kind. In particular, we proved several approximation results on these different spaces, some of them are based on constructive methods making use of the Taylor expansion and the convolution polynomials. The techniques used in these two cases are different. Moreover, for the second kind theory, we discussed also some density

results of reproducing kernels. This paper extends some classical results in complex analysis contained in the famous book of Kehe Zhu titled "Analysis on Fock spaces".

- The second paper is a joint work with my supervisor Prof. Sabadini and Prof. Alpay from Chapman University, US. In this article, we proposed a new definition extending to higher order the theory of slice hyper-holomorphic functions on the quaternions originally introduced by Gentili and Struppa in 2007. This definition extends the notion of complex polyanalytic functions to quaternions. We studied some basic properties of such functions and proved the counterparts of the following results: Splitting Lemma, Identity Principle, Representation Formula, Extension Lemma, Refined Splitting Lemma, Poly-decomposition and presented some of their consequences. Then, we considered the Fock and Bergman spaces in this new setting and computed explicit expressions of their reproducing kernels.
- The third paper is about a Cholewinski-Fock space in the slice hyperholomorphic setting. It studies an extension of the classical slice hyperholomorphic Fock space introduced in 2014 by Alpay, Colombo, Sabadini and Salomon. This was possible by considering on the space of slice entire functions a specific weight involving a modified Bessel function of the third kind, namely the Macdonald's function. We gave a complete description of this quaternionic Hilbert space. Then, its reproducing kernel is obtained making use of the slice hyperholomorphic extension of the classical complex Dunkl kernel. We introduced also an associated unitary integral transform, and study some specific quaternionic operators on the slice hyperholomorphic Cholewinski-Fock space. This construction follows an approach by Cholewinski in 1984.
- The fourth paper is a joint work with Prof. Sabadini and Prof. Krausshar from University of Erfurt, Germany. The results obtained in this research work can be considered as applications of the famous Sce-Qian-Fueter mapping theorem. It is well-known in the literature that this theorem relates two main theories in Clifford analysis, namely the recent theory of slice-monogenic functions and the classical one of monogenic functions (i.e: null solutions of Dirac operator). More precisely, making use of the Sce-Qian-Fueter mapping theorem we constructed and studied some special integral transforms of Bargmann-Fock type in the setting of quaternion slice hyperholomorphic and Cauchy-Fueter regular functions. In particular, starting with the normalized Hermite functions we got a Clifford-Appell system of quaternionic regular polynomials. We obtained also some new integral representations and generating functions in both the Fock and Bergman cases. In this article, we computed also the explicit expressions of the slice hyperholomorphic Bergman kernels on the quaternionic unit half ball and the fractional wedge domain.

- The fifth paper is a joint work with my colleague Antonino De Martino from Politecnico di Milano. In this paper, we introduced a new quaternionic short-time Fourier transform QSTFT with a Gaussian window. We proved several results about this QSTFT like a Moyal formula, a reconstruction formula and a Lieb uncertainty principle. This construction was possible thanks to the use of the quaternionic Segal-Bargmann transform. Moreover, we computed the reproducing kernel associated the Gabor space considered in this framework.
- The sixth paper is a joint work with Prof. Sabadini and Prof. Alpay. We proved here that slice polyanalytic functions of order n on quaternions can be considered as null solutions of the n -th power of a certain global operator with non-constant coefficients as it happens in the case of slice hyperholomorphic functions. We investigated also an extension version of the Fueter mapping theorem in this polyanalytic setting. In particular, we showed that under axially symmetric conditions it is always possible to construct Fueter regular and poly-Fueter regular functions through slice polyanalytic ones using what we call the poly-Fueter mappings. Furthermore, we proved a new poly-Cauchy formula that suggests to start several new interesting research problems. As a first application of this poly-Cauchy formula we gave an integral representation of the poly-Fueter mapping theorem, extending a very important result obtained in 2010 by Colombo, Sabadini, Sommen.
- The seventh paper is a joint work with Prof. Sabadini and Prof. Alpay. It deals with a specific system of Clifford-Appell polynomials and in particular their CK-product. We first studied how this system behave with respect to the CK-product. We gave also a characterization of Fueter hyperholomorphic functions of axial type in terms of this system. Then, we introduced a new family of quaternionic reproducing kernel Hilbert spaces in the framework of Fueter regular functions. This construction is based on a general idea which allows to obtain various function spaces, by specifying a suitable sequence of real numbers. We focused more on the Fock and Hardy cases and we studied the action of the Fueter mapping and its range.
- The eighth paper is a joint work with Prof. Sabadini, Prof. Colombo and Prof. Alpay. In this paper, we started a new research topic to begin the study of Schur analysis and de Branges-Rovnyak spaces in classical quaternionic and Clifford analysis, in particular in the framework of Fueter hyperholomorphic functions. We treated there several problems related to Hardy space, Schur multipliers, Blaschke functions, Herglotz multipliers and their associated kernels and Hilbert spaces.

4 Courses

I attended two PhD School courses, four courses from the department of mathematics at *Politecnico di Milano* and one summer school course at the *University of Bologna*. This allowed me to earn the 30 credits requested by the PhD Programme and pass the first and second year with a grade A. Here is the list of attended courses

4.1 PhD School Courses

1. *Epistemology of Scientific and Technical Research (passed, grade A)*. This course took place at *Politecnico di Milano* from 16 February 2018 to 02 March 2018. This course was organised by *Prof. Chiodo Simona* from *Politecnico di Milano*.
2. *Scientific Communication in English (passed, grade B)*. This course took place at *Politecnico di Milano* from 16 April 2018 to 04 May 2018. The lecturer of this course was *Prof. Tim Sluckin*, from the *University of Southampton*.

4.2 PhD Programme Courses

1. *Semi-linear Elliptic Equations (passed, grade A)*. This course took place at the mathematical department of *Politecnico di Milano* from 08 November 2017 to 31 January 2018. The lecturers of this course were *Prof. Gianmari Verzini* and *Prof. Soave Nicola* from *Politecnico di Milano*.
2. *Maximum Principles and Applications to Second Order Elliptic and Parabolic Partial Differential Equations (passed, grade A)*. This course took place at the mathematical department of *Politecnico di Milano* from 01 February 2018 to 31 March 2018. The lecturers of this course were *Prof. Monticelli Dario Daniele* and *Prof. Punzo Fabio* from *Politecnico di Milano*.
3. *Semi-groups of Linear Operators and Applications to Evolution Equations (passed, grade A)*. This course took place at the mathematical department of *Politecnico di Milano* from 15 March 2018 to 30 May 2018. The lecturers of this course were *Prof. Conti Monica* and *Prof. Pata Vittorino* from *Politecnico di Milano*.
4. *Generalizations of Holomorphic Functions and Applications (passed, grade A)*. This course took place at the mathematical department of *Politecnico di Milano* from 6 April 2020 to 4 May 2020. The lecturers of this course were *Prof. Sabadini Irene Maria* and *Prof. Stefano Pinton* from *Politecnico di Milano*.

4.3 Summer School Courses

1. *ACOTCA 2018: Advanced Courses in Operator Theory and Complex Analysis*, (passed, grade A), University of Bologna, June 11-14, 2018.

5 Scientific activities

5.1 Conferences, Workshops and Talks

During the period of this project I attended 13 international congresses, conferences and workshops in which I gave 8 talks. This list includes also two special events for Marie-Sklodowska Curie fellows that took place in Toulouse and Bari.

1. ACOTCA 2018, Advanced Courses in Operator Theory and Complex Analysis, Bologna University, Italy. June 11-14, 2018.
2. MSCA Satellite Event ESOF 2018, Toulouse, France. July 7-8, 2018.
3. International Conference on Hypercomplex Analysis and Applications, Erfurt University, Germany. September 24-26, 2018. Talk title: On slice polyanalytic functions of a quaternionic variable
4. Advances in Operator Theory with Applications to Mathematical Physics, Chapman University, California, US. November 12-16, 2018. Talk title: On slice polyanalytic functions of a quaternionic variable.
5. International Conference on Mathematical Methods and Physics, Marrakech, Morocco. April 1-5, 2019. Talk title: The Cholewinski-Fock space in the slice hyperholomorphic setting.
6. INdAM Day, Università di Bari, Italy. June 3, 2019.
7. 30th International Workshop on Operator Theory and its Applications, IWOTA 2019. Instituto Superior Técnico, Lisbon, Portugal. July 22-26, 2019.
8. 12th International ISAAC Congress, University of Aveiro, Portugal. July 29- August 2, 2019. Contributed talk title: On the Bargmann-Fock-Fueter and Bergman-Fueter integral transforms.
9. Quaternioni Sul Conero II, Università Politecnica delle Marche, Ancona, Italy. September 12- 13, 2019.

10. New Directions in Function Theory: From Complex to Hypercomplex to Non-Commutative, Chapman University, California, US. November 21-26, 2019. Talk title plan: Polynomial Approximation in slice regular Fock spaces.
11. International conference on Hypercomplex Analysis in Mathematics and in the Applied Sciences, Weimar, Germany. February, 19-21, 2020. Contributed talk title: A new poly-Fueter mapping theorem.
12. 12th International Conference on Clifford Algebras and Their Applications in Mathematical Physics (ICCA 2020), Hefei, China. August 03-07, 2020. Contributed talk title: On the global operator and Fueter mapping theorem for slice poly-analytic functions.
13. Recent progress in Hypercomplex analysis and geometry, Polimi, Milano, Italy. September 10-11, 2020. Contributed talk title: Fock and Hardy spaces: the Clifford-Appell case.

5.2 Research stay abroad

From 1 to 31 May 2019, I visited Chapman University in California, US. This gave me the opportunity to spend a part of my project abroad. In this research stay, I visited Prof. Dr, Daniel Alpay and Prof. Dr. Ahmed Sebbar with whom I had several interesting scientific discussions. Some parts of the project that are in progress thanks to this visit are related to the topics of, Wiener algebra on quaternions: Fueter variables case, some hyperholomorphic functional spaces, the Poly-Fueter mapping theorem and poly-Bergman-Fueter transforms. Moreover, as a part of this visit, I interacted also with some researchers working in Schmid college of Science and technology and gave a talk about "*Short-Time Fourier transform (SFT), Gabor and super Gabor spaces*" during *Prof. Alpay's seminar on Wavelets*.

5.3 Seminars

I attended several seminars and lectures that took place at *Politecnico di Milano, University of Bicocca, University of Milano* and Palazzo di Brera. This is a list of some selected lectures.

1. Spectral Theory, Sum Rules and Large Deviations, by *Prof. Barry Simon* from *California Institute of Technology*, 28 May 2018 at *Dipartimento di Matematica, University of Milano*.
2. Tales of Our Forefathers, by *Prof. Barry Simon* from *California Institute of Technology*, 29 May 2018, at *Dipartimento di Matematica, Politecnico di Milano*.

3. On the harmonicity of slice regular functions, by *Prof. Cinzia Bisi* from *Università di Ferrara*, 31 May 2018 at *Dipartimento di Matematica, Politecnico di Milano*.
4. The essential norm estimates of Hankel and the $\bar{\partial}$ -Neumann operators, by *Prof. Zeljko Cuckovic* from *University of Toledo*, 01 June 2018 at *Dipartimento di Matematica, University of Milano*.
5. Vaporizing and Freezing the Riemann Zeta Function, by *Prof. Terence Tao* from *University of California, Los Angeles*, 22 June 2018 at *University of Bicocca*.
6. Superoscillations and approximation of generalized functions, by *Prof. Daniele Struppa* from *Chapman University*, 10 June 2019 at *Dipartimento di Matematica, Politecnico di Milano*.
7. A new approach to quantum mechanics, by *Prof. Yakir Aharonov* from *Chapman University*, 18 June 2019 at *Dipartimento di Matematica, Politecnico di Milano*.
8. One Hundred Years of Universes, by *Prof. John Barrow* from *University of Cambridge*, 29 October 2019 at *Palazzo di Brera, Via Brera 28, Milano*.
9. On Mean Field Games, by *Prof. Pierre-Louis Lions* from *Collège de France*, 29 October 2019 at *Palazzo di Brera, Via Brera 28, Milano*.
10. On the Power of Geometric Illustration in Mathematics and Science, by *Prof. Roger Penrose* from *University of Oxford*, 29 October 2019 at *Palazzo di Brera, Via Brera 28, Milano*.

6 Publications

During this research project I wrote and co-authored 8 papers.

6.1 Original research articles

1. Alpay, D., Diki, K., Sabadini, I. On slice polyanalytic functions of a quaternionic variable. *Results Math.* 74 (2019), 25 pp.
2. Diki, K., Gal, S.G., Sabadini, I. Polynomial Approximation in slice regular Fock spaces. *Complex Anal. Oper. Theory.* 13 (2019), 18 pp.

3. Diki, K., Krausshar, R.S., Sabadini, I. On the Bargmann-Fock-Fueter and Bergman-Fueter integral transforms. *J. Math. Phys.* 60 (2019), 26 pp.
4. Diki, K. The Cholewinski-Fock space in the slice hyperholomorphic setting. *Math. Methods Appl. Sci.* 42 (2019), 18 pp.
5. De Martino, A., Diki, K. On the quaternionic short-time Fourier and Segal-Bargmann transforms. Accepted, to appear in *Mediterr. J. Math.* (2020), 23 pp.
6. Alpay, D., Diki, K., Sabadini, I. On the global operator and Fueter mapping theorem for slice poly-analytic functions. Accepted, to appear in *Anal. Appl.* (2020), 20 pp.
7. Alpay, D., Diki, K., Sabadini, I. Fock and Hardy spaces the Clifford-Appell case. Preprint, submitted (2020), 30 pp.
8. Alpay, D., Colombo, F., Diki, K., Sabadini, I. On a poly-analytic approach to noncommutative de Branges-Rovnyak spaces and Schur analysis. Preprint, submitted (2020), 54 pp.

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