Report on the second year as PhD Student

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Preprints

- *Uniform Matroids are Ehrhart positive*, Nov. 2019.
  https://arxiv.org/abs/1911.10146

  Abstract: De Loera et al. conjectured that the Ehrhart polynomial of the basis polytope of a matroid has positive coefficients. We prove this conjecture for all uniform matroids. In other words, we prove that every hypersimplex is Ehrhart positive. In order to do that, we introduce the notion of weighted Lah numbers and study some of their properties. Then we provide a formula for the coefficients of the Ehrhart polynomial of a hypersimplex in terms of these numbers.

- *On the Ehrhart polynomial of minimal matroids*, Apr. 2020.
  https://arxiv.org/abs/2003.02679

  Abstract: We provide a formula for the Ehrhart polynomial of the connected matroid of size n and rank k with the least number of bases, also known as a minimal matroid. We prove that their polytopes are Ehrhart positive and h^*-real-rooted (and hence unimodal). We use our formula for these Ehrhart polynomials to prove that the operation of circuit-hyperplane relaxation of a matroid preserves Ehrhart positivity. We state two conjectures: that indeed all matroids are h^*-real-rooted, and that the coefficients of the Ehrhart polynomial of a connected matroid of fixed rank and cardinality are bounded by those of the corresponding minimal matroid and the corresponding uniform matroid.

Schools, Conferences and Seminars

- *Einstein Workshop on Polytopes and Algebraic Geometry*
  Location: Freie Universität Berlin
  Date: December 1st - December 4th
  I attended several presentations by leading researchers in Discrete Geometry and Ehrhart Theory. This was also useful to establish social interactions with young researchers that also work in these areas.

I acted as a speaker in the following seminars.

- *BaD seminar* at the University of Bologna.
  Date: 16th December 2019
  Title of the presentation: *Ehrhart positivity of certain polytopes.*
- *Discrete Geometry Seminar* of the Freie Universität Berlin
  Date: May 7th 2020.
  Title of the presentation: *The Ehrhart Theory of Matroids*
- *PhD Students Seminar* of the Universidad Nacional de Córdoba
  Date: May 18th 2020
  Title of the presentation: *Enumerating the points inside of a polytope.*

I was also proposed to act as referee for the journal *Discrete and Computational Geometry* of Springer.

Research Activity

For a lattice polytope $\mathscr{P}$ in $\mathbb{R}^n$, one can consider the number of lattice points that lie inside the polytope, it is:

$$i(\mathscr{P}) \doteq |\mathscr{P} \cap \mathbb{Z}^n|.$$

Even more, for each *integer dilation* $k\mathscr{P}$ of $\mathscr{P}$, we can consider the number of points in $k\mathscr{P}$ and consider this as a function of k:

$$i(\mathscr{P}, k) \doteq i(k\mathscr{P}) = |k\mathscr{P} \cap \mathbb{Z}^n|.$$

In [4] Eugene Ehrhart proved this function is a polynomial in the variable k and since then its study grew and expanded, having applications in Order

Theory [14], Algebraic Geometry [], Number Theory [10], Algebraic Combinatorics [3], etc.

These polynomials are what now in the mathematical community people know as the *Ehrhart polynomial* of a polytope.

A major (and thus far, intractable) problem is to classify all the polynomials that arise as the Ehrhart polynomial of a lattice polytope. This could be done only for 2-dimensional polytopes (aka *polygons*) [13].

If $\mathscr{P}$ is an m-dimensional lattice polytope in $\mathbb{R}^n$, one can show that its Ehrhart polynomial has the form:

$$i(\mathscr{P}, t) = a_m t^m + a_{m-1} t^{m-1} + a_{m-2} t^{m-2} + \ldots + a_1 t + a_0.$$

Where:

- a_m = (normalized) volume of $\mathscr{P}$
- a_{m-1} = half the sum of the (normalized) volume of the facets of $\mathscr{P}$
- $a_0 = 1$.
- Nothing can be said of $a_1, \ldots, a_{m-2}$. They can be negative [9].

In [2] it was conjectured that a *matroid polytope* has an Ehrhart polynomial with positive coefficients. These are defined as follows:

Given a matroid $M = (E, \mathscr{B})$ where $|E| = n$, and $\mathscr{B}$ is the set of bases,

$$\mathscr{P}(M) = \text{convex hull}\{e_B : B \in \mathcal{B}\},$$

where $e_B \doteq \sum_{i \in B} e_i$ and each e_i is a canonical vector in $\mathbb{R}^n$. These polytopes are central objects in combinatorics. A geometric characterization of them was found in [6].

When De Loera - Haws - Köppe posed the Ehrhart positivity conjecture for matroids in [2], they gave a proof for all uniform matroids of the form $U_{2,n}$ (n elements and rank 2).

In [5] I was able to extend this result to all uniform matroids $U_{k,n}$ for arbitrary k and n. This is the first example of an infinite family of matroids of all ranks and cardinalities having that property.

The polytope of the uniform matroid $U_{k,n}$ is by itself an interesting object of study in combinatorics. One can prove that:

$$\mathscr{P}(U_{k,n}) = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \right\}$$

This polytope is known as the (n,k)-*hypersimplex*. So our result can be rephrased to say *all hypersimplices are Ehrhart positive*. The Ehrhart polynomials of hypersimplices have been extensively studied.

We gave a prove that the coefficients of these Ehrhart polynomials are positive by introducing a new statistic on linear ordered partitions, which we call *weighted Lah numbers*.

For each n and k one can also consider the *smallest* (connected) matroid with n elements and rank k. In a second article we were able to prove that these matroids are Ehrhart positive. Also, we refined the Conjectures posed in [2] and verified computationally that all matroids with up to 9 elements have an Ehrhart polynomial that is coefficient-wise bigger than that of the corresponding minimal matroid.

Moreover, if M is a matroid that has a hyperplane H that is also a circuit, one can define a new matroid $\widetilde{M}$ on the same groundset of M, that has the same set of bases of M and an extra basis: H. We were able to prove that this operation preserves Ehrhart positivity, and moreover, that geometrically it is just gluing the polytope of a minimal matroid on a facet. Our results also permit to construct non-isomorphic and non-dual connected matroids with the same Tutte polynomial and the same Ehrhart polynomial.

Future Research

Recently I was able to extend the result of the Ehrhart positivity of hypersimplices, using properties of weighted Lah numbers, to the family of half-open hypersimplices [7]:

$$\Delta'_{k,n} = \left\{ x \in [0,1]^{n-1} : k-1 < \sum_{i=1}^{n} x_i \leq k \right\},$$

This seems to be a useful tool to settle the Ehrhart positivity of many other polytopes, since we can consider (disjoint) tilings using half-open hypersimplices.

For example, the truncated-cube:

$$\square'_{k,n} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq k \right\},$$

can be tiled using half-open hypersimplices that do not overlap, and therefore its Ehrhart polynomial is the sum of their Ehrhart polynomials and thus has positive coefficients.

There are some subfamilies of Lipschitz polytopes [12] and flow polytopes that can be attacked with this approach. Moreover, the truncated cube $\square'_{k,n}$ is the *independence matroid polytope* (IMP) of the uniform matroid $U_{k,n}$. So now we have a proof that uniform matroids have an Ehrhart positive IMP!

Very little is known about the Ehrhart theory of independence matroid polytopes. I recently noticed that they are integrally equivalent to a generalized-permutohedron [11], so they are candidates to be Ehrhart positive [8].

My plan is to explore what happens with the Ehrhart polynomial of independence matroid polytopes. In particular, studying *generalized Catalan matroids* [1] I was able to prove that their polytopes are the intersection of a hypercube $[0,1]^n$ and a Stanley-Pitman polytope, and since the latter are a Ehrhart positive family, we could try to adapt the proof of that fact for these IMPs.

On the other hand, a relevant (more general) problem is Ehrhart positivity for *generalized permutohedra*, a wider class of polytopes. Among these polytopes one may consider the Minkowski sum of dilated hypersimplices:

$$t_1\Delta_{n,1} + t_2\Delta_{n,2} + \ldots + t_{n-1}\Delta_{n-1,n}.$$

For $t_1, \ldots, t_{n-1} \geq 0$ the family of polytopes one can obtain this way are what is known as *generic permutohedra*. It is an open problem to understand the Ehrhart coefficients for these. I want to explore if my result for hypersimplices can be used to attack that new family.

References

[1] Joseph Bonin, Anna de Mier, and Marc Noy. Lattice path matroids: enumerative aspects and tutte polynomials. *Journal of Combinatorial Theory, Series A*, 104(1):63 – 94, 2003.

[2] Jesús A. De Loera, David C. Haws, and Matthias Köppe. Ehrhart polynomials of matroid polytopes and polymatroids. *Discrete Comput. Geom.*, 42(4):670–702, 2009.

[3] Michele D'Adderio and Luca Moci. Ehrhart polynomial and arithmetic Tutte polynomial. *European Journal of Combinatorics*, 33(7):1479 – 1483, 2012. Groups, Graphs, and Languages.

[4] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris*, 254:616–618, 1962.

[5] Luis Ferroni. Uniform Matroids are Ehrhart Positive, https://arxiv.org/abs/1911.10146. *Submitted*, 2020.

[6] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. *Combinatorial geometries, convex polyhedra, and Schubert cells*, volume 63. 1987.

[7] Nan Li. Ehrhart h^*-vectors of hypersimplices. *Discrete Comput. Geom.*, 48(4):847–878, December 2012.

[8] Fu Liu. On positivity of Ehrhart polynomials. In *Recent trends in algebraic combinatorics*, volume 16 of *Assoc. Women Math. Ser.*, pages 189–237. Springer, Cham, 2019.

[9] Fu Liu and Akiyoshi Tsuchiya. Stanley's non-ehrhart-positive order polytopes. *Advances in Applied Mathematics*, 108:1–10, Jul 2019.

[10] H. Mohades and B. Honari. On a relation between spectral theory of lens spaces and ehrhart theory, 2016.

[11] Alexander Postnikov. Permutohedra, Associahedra, and Beyond. *International Mathematics Research Notices*, 2009(6):1026–1106, 01 2009.

[12] Raman Sanyal and Christian Stump. Lipschitz polytopes of posets and permutation statistics. *Journal of Combinatorial Theory, Series A*, 158:605 – 620, 2018.

[13] P.R. Scott. On convex lattice polygons. *Bulletin of the Australian Mathematical Society*, 15(3):395–399, 1976.

[14] Richard Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1:9–23, 1986.