REPORT ON SCIENTIFIC ACTIVITIES (2019-2020)

Dottorato Università degli Studi di Genova-XXXIV cycle

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1. Research project and research activities

My research project concerns the study of rational maps between projective spaces. I'm especially interested in classifying birational maps which are the rational maps inducing an isomorphism between dense open sets of projective spaces.

• **Sum up of 2018-2019.** During my first year at the Dipartimento di Matematica of Genova (DIMA), I focused on the so-called *gradient maps* which are the rational maps defined by the partial derivatives of an initial polynomial. Using techniques of reductions modulo p, I described in particular birational gradient maps of arbitrarily large algebraic degree over fields of arbitrarily large characteristic. This work was subject to a pre-publication on Arxiv [BC19] and is currently submitted in a journal.

In parallel to this work and under the influence of Alessandro de Stefani from DIMA, I get acquainted with the more algebraic approach of singularities via the description of the *F-signature* of a ring which encodes the behaviour of the Frobenius morphism. In practical, this first contact materialized in reading classical books on the subject and studying research articles. Though I'm still struggling with technicalities, I'm now capable of following research talks in this area, talks I attend in particular in the *Virtual Commutative Algebra seminars* of the department of Mathematics of the IIT Bombay (India) since April 2020.

• **Determinantal maps and their mixed multiplicity (2019-2020).** Among birational maps, some are defined by the maximal minors of a defining inital matrix with polynomial entries, the so-called *determinantal maps*, and, as such, share special features concerning the computation of their mixed multiplicity. Recall that given a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, the *mixed multiciply* $e_f(\mathbf{n}) = \big(e_f(n,0), \ldots, e_f(0,n)\big)$ of f is the $(n+1)$-uple such that for any $i \in \{0, \ldots, n\}$, $e_f(n-i,i) = \#\big(H_{\mathbf{x}}^{n-i} \cap f^{-1}(H_{\mathbf{y}}^i)\big)$ is the cardinal of the intersection $H_{\mathbf{x}}^{n-i} \cap f^{-1}(H_{\mathbf{y}}^i)$ of a codimension $n-i$ general linear space $H_{\mathbf{x}}^{n-i}$ in the source space of f with the preimage $f^{-1}(H_{\mathbf{y}}^i)$ of a codimension i general linear space $H_{\mathbf{y}}^i$ in the target space of f (a birational map f being a map such that $e_f(0,n) = 1$).

During 2019-2020, my investigations on this subject was structured around three axes:

(i) **Find examples of determinantal birational maps.** This axis was mainly experimental but lead to describe determinantal birational maps as the solutions of interpolations of the entries of their defining matrix. Actually, this prospecting phase results in answering negatively a structuring question of this project (though the answer follows from elementary counter examples):

> **Problem A.**
>
> Is the inverse of a determinantal birational map determinantal?

(ii) **Give a combinatorial description of determinantal birational maps.** The underlying motivation of this latter problem was to obtain a similar description of determinantal birational maps as it can be given for monomial birational maps (see for instance [Cor14]). The most satisfactory combinatorial description I could produce until now is a translation of the computation of the mixed multiplicity in the computation of the mixed volumes of the polytopes associated to the defining matrix of a determinantal map. It actually gives a fruitful point of view because, first, it is specific to determinantal maps and, second, it leads to construct determinantal birational maps via basic formulas in Convex geometry. It gives additionally an original perspective on the main examples of determinantal maps described in the litterature over the past years, for instance in [PGS06] and [DH17].

(iii) **Describe the equations of the graphs of determinantal Cremona maps.** The graph of a map being the Proj of the Rees algebra of the base ideal of the map, this problem is nothing but the very classical problem consisting in describing the generators of the Rees algebra of an ideal starting from the generators of the symmetric algebra of the ideal, see next subsection for more about this topic.

I'm currently writing a sum up of my research concerning Item (i) and Item (ii) since I consider I cannot say much more about these two first axis. Though I'm still not sure this writing can be subject to a publication, I believe it's valuable to have a concise and definitive overview of this past work, especially to focus on the generators of the Rees algebras of determinantal maps for the year to come.

• **Generators of the Rees algebras of determinantal maps (2019-2021).** Regarding birational maps $f = (f_0 : \dots : f_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, the equations of the Rees algebras of the base ideal $\mathcal{I}_f = (f_0, \dots, f_n)$ of f are of special interest because they describe in particular the base ideal $\mathcal{I}_{f^{-1}}$ of the inverse map $f^{-1} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of f. Actually, the equations of the Rees algebras are of interest in a way broader context than birational maps and this topic is consequently the object of a numerous and dynamic literature.

In last February-March 2020, I had the chance to visit Professor Bernd Ulrich and Professor Claudia Polini, both experts in this topic, in Purdue University, West Lafayette (USA) where I get acquainted with parts of this "Rees algebras literature". This visit also lead to describe the generators of the Rees algebras of determinantal birational maps of $\mathbb{P}^3$ whose defining matrix is almost linear, see for instance [KPU11] for references about almost linear matrix. This classification was actually very surprising since it revealed two noteworthy ideals of Rees algebras of determinantal birational maps of $\mathbb{P}^3$ when we might have expected only one.

For the year to come, I'm planning to focus on describing the generators of the Rees algebras of the families of the determinantal maps I could characterize until now (see Item (i) and Item (ii) of the previous subsection). Since I believe this attempt could also interest Professor Bernd Ulrich and Professor Claudia Polini, I'm currently rushing into finishing my writing about Item (i) and Item (ii) of the previous subsection in order to present them this delimited project that we might deal together.

• **Kozsul property of generic projections and Gorenstein height 3 birational maps (2020-2021).** Since july 2020, I get acquainted by Profesor Aldo Conca to the subject of the Koszul property of projections of Veronese variety from a generic point. The goal is to show that the coordinate rings of these projections share the property of being *Koszul*, i.e. that the free resolution of their coordinate ring is linear (even if infinite). In practice, I get familiarized with the article [CC13] and the reference therein.

From a more down-to-earth perspective, this subject made also me think to the birational maps whose base ideal is Gorenstein of height 3. In the end, this is just a similar property of being determinantal but seems not to have been considered from the perspective of birational map in the literature. In particular, I'm currently wondering if the computation of the characteristic classes of general Gorenstein height 3 birational maps could be carried out in the same way it was down for general determinantal birational maps in [PGS06]. I hope I'll have time to deepen this question for the year to come.

2. Courses, seminar, workshop attended

- 13-17/01/2020, *9th swiss-french workshop in Algebraic Geometry* in Charmey (Switzerland).
- 03-06/2019: e-reading course in Commutative Algebra of the DIMA.
- 04/2020-present: *Virtual Commutative Algebra seminars* of the department of Mathematics of the IIT Bombay (India).
- 10-11/2020: weekly *Working group in Mori's theory* of the DIMA.

3. Visiting period to other universities

- 21/02-13/03/2020, *visiting periode at Purdue University (West Lafayette, Indiana)* of Professors Bernd Ulrich and Claudia Polini.

4. Scientific diffusion

- 10/2019: talk *Transformations polaires birationnelles du plan de degré arbitrairement grand* in the seminar of the team Géométrie, Algèbre, Dynamique et Topologie of the Institut Mathématique de Bourgogne, Dijon (France).
- 03/2020: talk *Homaloidal curves of arbitrarily large degree in positive charateristic*, in the Commutative Algebra Seminar of the mathematic departement of Purdue' University, West Lafayette (USA).
- 05/2020: two talks *Integral closure: examples and applications* in the e-reading course in Commutative Algebra of the DIMA.

5. Paper submitted

[BC19] *Plane polar cremona maps of arbitrarily large degree in positive characteristic*, 9 pages, available at arXiv:1910.0120.

References

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[KPU11] A. Kustin, C. Polini, and B. Ulrich. Rational normal scrolls and the defining equations of Rees algebras. *J. reine angew. Math.*, 650:23–65, 2011.

[PGS06] I. Pan and G. Gonzalez Sprinberg. On characteristic classes of determinantal Cremona transformations. *Mathematische Annalen*, 335:479–487, 2006.

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