Relation associated with the research activity during the first year of the Ph.D. program

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Cycle: XXXIV

1. Attended courses and seminars

In the academic year 2018-2019 starting from the month of November I have attended the following courses dedicated for the first year Ph.D. students:

1) *Functional Analysis and Complex Analysis*,  
   held by Prof. Nicola Arcozzi and Pavel Mozolyako

2) *Introduction to Cartan geometry and its applications*,  
   held by Prof. Emanuele Latini and Prof. Andrea Santi

3) *Convex Optimization for Imaging*,  
   held by Prof. Alessandro Lanza

4) *Topics in global analysis*,  
   held by Prof. Gerardo A. Mendoza

5) *The de Branges theory of Hilbert spaces of entire functions and its applications to spectral theory of differential operators*,  
   held by Prof. Anton Baranov.

Those courses had on average the length of 14 hours each and concluded with a final examination to evaluate the good knowledge of treated course contents. Besides the courses stated above I took part in the cycle of seminars *Topics in Mathematics* organized by Prof. Giovanni Mongardi and Prof. Giovanni Cupini that covered different mathematical research areas and didactics. I also frequented multiple seminars held in the department and outside among which Pini seminars and analysis seminars.

2. Schools, conferences and missions abroad

- *Winter School – Stochastic PDEs and Mean-Field Games*  
  Place: University of Bologna  
  Period: 14th of February 2019 - 16th of February 2019  
  The winter school was divided in two minicourses, the first presented a review of the compactness method for solving the stochastic differential equations, the second introduced optimal control problems and mean-field games, minicourses were complemented by various talks related to SDEs.
• *Indam day*
  Place: University of Bari  
  Period: 3th of June 2019  
  This conference day was mainly dedicated to the currently actual mathematical topics among which talks in stochastic and numerical calculus. Besides this, various initiatives were made to connect diverse researchers.

• *Partial differential equations from theory to applications*
  Place: The University of Milan  
  Period: 1st of July 2019 - 5th of July 2019  
  The program of this summer school addresses recent topics related to partial differential equations, free boundary problems and integro-differential equations of fractional type, both from a rigorous theoretical perspective and in view of concrete real-world applications.

• *Research trip at the UWA*
  Place: The University of Western Australia, Perth  
  Period: 27th of August 2019 - 11th of October 2019  
  During the stay at Perth I attended courses on differential geometry and partial differential equations. On the 2nd of October I took part in the poster session in the annual Engineering and Mathematical Sciences Higher Degree by Research Conference.

3. Research activity

In the beginning of my doctorate program I have been introduced to the framework of non-commutative groups, in particular, the *Heisenberg group* which is the simplest Carnot group. I have studied (see [3]) the notions of $v$-convexity, horizontal gradient and intrinsic Hessian matrix which could be useful to generalize some classic results. One of the possible applications is considering the set of minimizers of a certain functional appeared in the article of H. W. Alt, L. A. Caffarelli and A. Friedman in 1984 (see [1]). In particular, we are interested in functionals like

$$ F(u) = \int_\Omega (||\nabla_{H^1} u||^2 + \chi_{\{u>0\}} + 2fu), $$

where $\Omega \subset H^1$, $H^1$ denotes the Heisenberg group, $\nabla_{H^1} u$ is the horizontal gradient in the Heisenberg group, $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$ and $f$ is a given function. The main goal is to discover the properties that critical points of such a functional have. Functionals of that type are crucial in variational problems with two phases.

Then my research work was related to the study of the regularity properties of the boundary of an open set $\Omega$ involved in the *Dirichlet problem for the fractional Laplace* operator. The Dirichlet problem for the fractional Laplacian consists in finding an unknown function $u$ that satisfies $(−\Delta)^s u = 0$ in $\Omega$ and is equal to a given function $g$ in $\Omega^c$, where $s \in (0, 1)$ fixed. Although the fractional Laplacian proves to be the usual Laplace operator when $s \rightarrow 1$ (refer to [5]), these two operators are structurally different due to the nonlocal nature of the fractional
Laplacian and one should consider the behaviour of $u$ at infinity. It is well known that the Dirichlet problem in the ball in the nonlocal framework is solved thanks to the existence of the Green function. Naturally arises the question of solving this problem in an arbitrary open set $\Omega$ using the Perron method.

During the first year I have been studying the necessary techniques for dealing with the above stated problem. At first, I studied the classical local case. Then I faced the definitions and properties of the Poisson kernel, the Fundamental solution and the Green function in the nonlocal case (see [2]) at the same time trying to analize the feasible definitions of harmonic lifting. Later I passed to the studying of the Kelvin transform and its properties considering two cases: when the inversion is centered at a point $x_0 \in B_r(0)$ or at $x_0 \in B^c_r(0)$, this is a useful tool for dealing with the Poisson kernel (see [6]).

In the last few months one of my research interest is the new notion of the divergent fractional Laplacian that was introduced by S. Dipierro, O. Savin and E. Valdinoci in 2016 (see [4]). In the paper the authors consider the new weaker condition on the behaviour of functions on infinity so that one could take the fractional Laplacian of such a function. One could ask if it is possible to consider the integro-differential operators of different type than the fractional Laplacian. Also arise natural questions about the existence of Schauder estimates, Liouville-type results or the solution for the Dirichlet problem with such imposed conditions.

References


