INdAM-COFUND-2012

INdAM Outgoing Fellowships in Mathematics
and/or Applications
cofunded by Marie Curie actions

MPoisCoho

Poisson cohomology
of multidimensional Hamiltonian structures

Final report

Period covered	from 01/10/2016 to 30/04/2018
Start date of the project	01/10/2016
Date of the report	11/06/2018
Fellow	dr. Matteo Casati
Personal supervisor in the Outgoing phase	prof. Evgeny Ferapontov
Host organisation for the Outgoing phase	Loughborough University
Personal supervisor in the Return phase	prof. Boris Dubrovin
Host organisation for the Outgoing phase	Scuola Internazionale Superiore di Studi Avanzati di Trieste

Table of contents

1 Final Publishable Summary Report

1.1 Executive summary

The project MPoisCoho addressed the computation of the Poisson Cohomology for multidimensional Hamiltonian structures, namely of the Hamiltonian structure of PDE for functions of several space variables. Indeed, only in the recent years investigations about this higher dimensional case has received wider attention.

One of the main reasons for this fact are the technical difficulties that arise in most of the computations, together with the existence of only partial results in the classification of the structures themselves. We exploited the notion of multidimensional Poisson vertex algebra to carry out explicit computations for the (already classified) two dimensional structures, which led to the discovery of the first nontrivial cocycles for some bidimensional structure and – in one case – an explicit new biHamiltonian structure. Moreover, an approach based on modern homological algebra has been followed to draw more general conclusions about the Poisson cohomology of higher dimensional structures, which is related to the theory of their deformations.

This study has applications in the theory of Integrable Systems, in particular the study of Integrable PDEs and nonlinear wave equations.

Despite the early termination of the project (due to a new appointment of the fellow), most of the objectives of the project have been achieved. Now we have a clearer understanding of the structure of the lowest cohomology groups for multidimensional brackets of hydrodynamic type, with explicit examples of cocycles and of a novel biHamiltonian structure. Moreover, the general form for the dispersive deformations of a scalar two-dimensional structure of hydrodynamic type has been obtained. This latest result and the techniques used to obtain it allow, for the months and years to come, to extend this line of research towards the Poisson cohomology of general (namely, of higher order) Hamiltonian structures. As a consequence, new integrable PDEs may be found.

1.2 Project objectives

The original objectives of this project were

1. Explicitly compute the lower cohomology groups for $(d = 2, n > 2)$, $(d = 3, n = 3)$ Poisson brackets of hydrodnamic type not studied yet, starting from the existing classification by Ferapontov and collaborators;

2. Find a general theorem for the Poisson cohomology of constant Hamiltonian structures of hydrodynamic type;

3. Adapt the homological algebraic method used in the constant case to the linear one.

During the project, two more objectives have been identified

4. Study the theory of deformation of multidimensional Poisson brackets of hydrodynamic type to characterise their normal form up to Miura transformations;

5. Contribute to the classification of higher order weakly nonlocal Poisson brackets of hydrodynamic type.

The overall aim of this project was to deepen our understanding of multidimensional Hamiltonian structures, working towards the establishment of normal forms for such differential operators. The combined approach of computer computation and of differential geometric and algebraic techniques, as the one we have devised, has significantly consolidated the already achieved results and paved the road to a broad classification.

1.3 Summary of results

Objective #1 The computer software developed to address the computation has been optimised and made faster and more reliable.

The fellow has computed the third degree component of H^2 for the $n = d = 2$ Poisson brackets of hydrodynamic type and proved that H^3 is not trivial as well, drawing partial conclusions in the direction of finding the normal form for $n = d = 2$ Hamiltonian operators.

Objective #2 The theorem proved in the scalar, multidimensional case [1] has been extended to a class of n component operators, in particular including an explicit description of all the cohomology classes for one of the three normal forms of the two-component Poisson bracket of hydrodynamic type.

Objective #3 Partial results obtained while working on Objective #4 have laid the basis of a broader project, conducted together with G. Carlet, about the extension of Getzler's theorem for $d = 1$, in which the particular form of the undeformed bracket (of first order in this case) it is not essential for the computation of the Poisson cohomology, hence being able to include the linear ones. For this reason – and because of the early termination of the project – this objective is left for future investigations.

[1] See Publication [1]

Objective #4 The $(d = 2, n = 1)$ case (for which in Publication [1] we obtained the full Poisson cohomology) has been completely solved – the only nontrivial dispersive deformations of the bracket of hydrodynamic type being classified by an infinite sequence of real constants. The results are reported in Publication [2].

The opening of this – not anticipated – line of research caused a shift of our interest from the explicit computation of the cohomology groups to the homological algebraic approach to the problem. For such reason we did not performed the (resource-intensive) computations needed in the $(d = n = 3)$ case.

Objective #5 In collaboration with his supervisor for the Outgoing phase and other researchers, the fellow has contributed to Publication [5], in which the classification of weakly non-local, third order Hamiltonian operators for $(d = 1, n > 2, 3)$ systems has been obtained. Moreover, the set of conditions for such operators in the case $n > 1$ has been obtained. After acknowledging the effectiveness of the theory of nonlocal Poisson vertex algebras[2], we have been able to exploit its axioms to compute the Hamiltonian conditions (the main contribution by the fellow in the paper). Their geometric interpretation has allowed us to perform a full classification for the lower dimensional cases of $n = 2$ and $n = 3$.

[2]see De Sole, A. & Kac, V.G. Jpn. J. Math. (2013) 8: 233. `https://doi.org/10.1007/s11537-013-1306-z`

2 Summary of the activities

2.1 List of Publications

No.	Title	Author(s)	Journal	Number	Pages	Status
1	Poisson cohomology of scalar multidimensional Dubrovin-Novikov brackets	G. Carlet, MC, S. Shadrin	J. Geom. Phys.	114 (2017)	404–419	Peer reviewed, published. Open access to preprint arXiv/1512.05744
2	Normal forms of dispersive scalar Poisson brackets with two independent variables	G. Carlet, MC, S. Shadrin	Letters in Math. Phys.	2018, on-line	1–25	Peer reviewed, published. Open access DOI 10.1007/s11005-018-1076-x
3	Higher order dispersive deformations of multidimensional Poisson brackets of hydrdoynamic type	MC	Theoret. Math. Phys.	in press		Peer reviewed, accepted for publication. Open access to preprint arXiv/1710.08175
4	MasterPVA and WAlg: Mathematica packages for Poisson vertex algebras and classical affine W-algebras	MC, D. Valeri	Bullett. Unione Matematica Italiana	2017, on-line		Peer reviewed. Preprint arXiv/1603.05028
5	On a class of third-order nonlocal Hamiltonian operators	MC, E. Ferapontov, M. Pavlov, R. Vitolo				Submitted. Preprint arXiv/1805.00746

2.2 List of dissemination activities

No.	Type	Title	Date/Period	Place	Audience
1	Seminar	Multidimensional Hamiltonian operators and Poisson vertex algebras	November 2016	Canterbury, UoK	Researchers
2	Seminar	Algebraic methods for the theory of deformation of multidimensional Hamiltonian operators	February 2017	University of Leeds	Researchers
3	Seminar	Dispersionless limit for the bi-Hamiltonian strcture of classical affine W-algebras: scouting for algebraic Frobenius manifolds	March 2017	University of Glasgow	Researchers
4	Communication at conference	Scalar multidimensional Hamiltonian operators of hydrodynamic types	May 2017	Montecatini Terme, National conference of GNFM	Researchers
5	Seminar at conference	Multidimensional Hamiltonian operators and Poisson vertex algebras	November 2016	Canterbury, UoK	Researchers
6	Seminar at conference	Normal form of dispersive scalar Poisson brackets for two-dimensional systems	June 2017	Gallipoli (LE)	Researchers
7	Seminar	Algebraic methods for the theory of deformation of multidimensional Hamiltonian operators	October 2017	University of Padova	Researchers
8	Seminar	Nonlocal Poisson vertex algebras and nonlocal Hamiltonian operators of third order	December 2017	University of Bourgogne-Franche-Comté, Dijon	Researchers
9	Seminar	Poisson brakets of hydrodynamic type and Poisson vertex algebras	January 2018	UEA, Norwich	Researchers
10	Presentation	Star polygons. An insight into medieval geometry	April 2018	SISSA, Trieste	School students

2.3 Other activities conducted by the fellow

2.3.1 Short visits

1. Korteweg-De Vries Institut, Universiteit Van Amsterdam. December 11th–15th, 2016. Hosts: Sergey Shadrin and Guido Carlet.
2. Dipartimento di Matematica e Applicazioni, University of Milan–Bicocca. January 24th–26th, 2017. Host: Paolo Lorenzoni.
3. Université de Bourgogne-Franche-Comté in Dijon (F). December 11th–15th, 2017. Host: Guido Carlet

2.3.2 Visitors

1. Guido Carlet. May 22nd-26th.

Workshop, conferences and schools attended

1. *Integrable Day*, November 25th, 2016. Workshop at Loughborough University.
2. *Schrödinger Equation: asymptotics, integrability and beyond.* Workshop at University of Bristol, March 28th, 2017.
3. National meeting of INdAM-GNFM, Montecatini (PT, Italy), May 4th–6th, 2017.
4. *INdAM Day*, Messina, June 7th, 2017.
5. *Schrödinger Equation: asymptotics, integrability and beyond.* Workshop at Loughborough University, May 16th 2017.
6. *Physics and Mathematics of Nonlinear phenomena 2017 - 50 years of IST*. International conference at Gallipoli (LE, Italy). June 17th–24th, 2017.
7. *Discrete models in Geometry and Physics*, Summer School. TU Berlin, September 11th–18th, 2017.
8. *Modern geometry and physics*, Workshop. Loughborough University, 18th–21st September, 2017.
9. *INDaM-Cofund day*, Workshop and training for Marie Curie fellows of INdAM. Rome, 4th June, 2018.