Approssimazione dello spettro di operatori di evoluzione per equazioni con ritardo neutrali lineari tramite metodi pseudospettrali

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The problem

Goal: stability of periodic solution of delay equations

Applications: control theory, population models, lasers, ...

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Applications: control theory, population models, lasers, ...

Technique: pseudospectral collocation $\rightsquigarrow$ approximating matrix

Delay equations

X state space, a function space on $[-\tau, 0]$

$$x_t(\theta) \coloneqq x(t+\theta), \quad \theta \in [-\tau, 0]$$

delay differential eq. (DDE)

$$x'(t) = h(x_t)$$

renewal eq. (RE)

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DDE $\quad x'(t) = f\left(x(t), x(t-\tau_i), \int_{-\tau}^{0} g(x(t+\theta))\, d\theta\right)$

RE $\quad x(t) = f\left(\int_{-\tau}^{0} g(x(t+\theta))\, d\theta\right)$

NDDE $\quad x'(t) = f\left(x(t), x(t-\tau_i), \int_{-\tau}^{0} g(x(t+\theta))\, d\theta, x'(t-\tau_i)\right)$

NRE $\quad x(t) = f\left(x(t-\tau_i), \int_{-\tau}^{0} g(x(t+\theta))\, d\theta\right)$

(N = neutral)

Theory: linearised stability and Floquet theory

ODE	✓	well-known
DDE	✓	[2, 3]
NDDE	✓	[3]
RE	✓	[2, 4]
NRE	✗	requires new perturbation theory [5]

[2] Diekmann, Van Gils, Verduyn Lunel, and Walther, *Delay Equations. Functional-, Complex-, and Nonlinear Analysis*, Appl. Math. Sci. 110, Springer, New York, 1995, DOI:10.1007/978-1-4612-4206-2.

[3] Hale, *Theory of Functional Differential Equations*, Appl. Math. Sci. 3, Springer, New York, 1977, DOI:10.1007/978-1-4612-9892-2.

[4] Breda and Liessi, *Floquet theory and stability of periodic solutions of renewal equations*, J. Dynam. Differential Equations (2020), DOI:10.1007/s10884-020-09826-7.

[5] Diekmann and Verduyn Lunel, *Twin semigroups and delay equations*, arXiv:1906.03409 (2019).

The monodromy operator of a linear periodic equation

IVP for DDE

$$\begin{cases} x'(t) = L(t)x_t \\ x_0 = \varphi \end{cases}$$

$$L(t)x_t = A(t)x(t) + \sum B_i(t)x(t-\tau_i) + \int_{-\tau}^{0} C(t,\theta)x(t+\theta)\,d\theta$$

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reformulation

$$T\varphi = V(\varphi, w)_p$$
$$w = \mathcal{F}V(\varphi, w)$$

$$(\mathcal{F}u)(t) = L(t)u_t$$

$$V(\varphi, w)(t) = \dots$$

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$$\dots = \begin{cases} \varphi(0) + \int_0^t w(\sigma)\,d\sigma, & t \in (0,p] \\ \varphi(t), & t \in [-\tau, 0] \end{cases}$$

DDE

$$\dots = \begin{cases} w(t), & t \in (0,h] \\ \varphi(t), & t \in [-\tau, 0] \end{cases}$$

RE

The monodromy operator of a linear periodic equation

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IVP for NRE

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$$L(t)x_t = \int_{-\tau}^{0} C(t,\theta)x(t+\theta)\,d\theta + \sum B_i(t)x(t-\tau_i)$$

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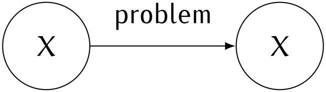
NRE

Pseudospectral methods



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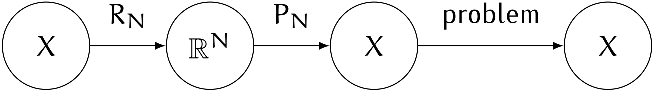

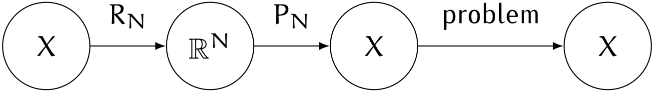

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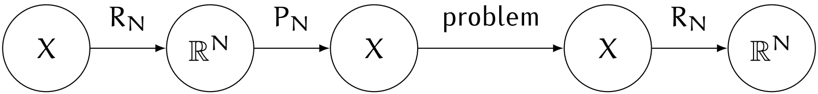

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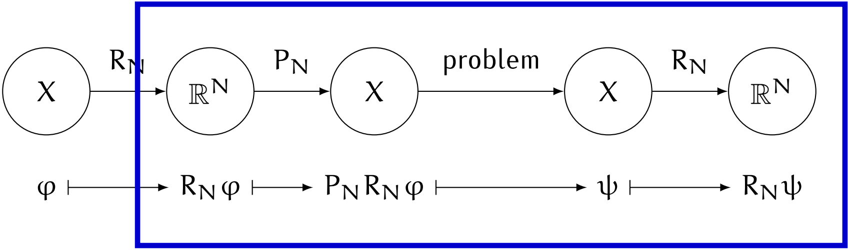

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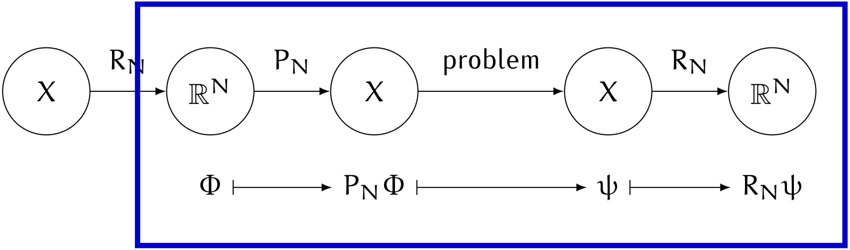

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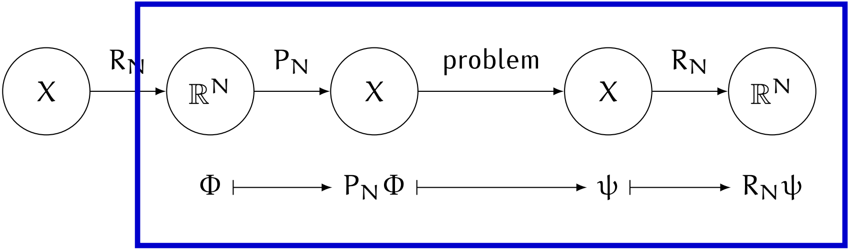

$\rightsquigarrow$ spectral accuracy [6]

with "good" nodes (e.g., Chebyshev zeros or extrema)
smooth functions: error $= O(N^{-k})$ for every k
analytic functions: error $= O(c^N)$ for $0 < c < 1$

[6] Trefethen, *Spectral Methods in MATLAB*, Software Environ. Tools, SIAM, Philadelphia, 2000, DOI:10.1137/1.9780898719598.

Pseudospectral collocation of the monodromy operator

- $M+1$ Chebyshev extrema in $[-\tau, 0]$, R_M, P_M
- N Chebyshev zeros in $[0, p]$, R_N^+, P_N^+
- Discretize $T\colon X \to X$

$$\begin{aligned} T\varphi &= V(\varphi, w)_p \\ w &= \mathcal{F}V(\varphi, w) \end{aligned}$$

as $T_{M,N}\colon \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$

$$\begin{aligned} T_{M,N}\Phi &= R_M V(P_M\Phi, P_N^+ W)_p \\ W &= R_N^+ \mathcal{F}V(P_M\Phi, P_N^+ W) \end{aligned}$$

- Compute the eigenvalues of $T_{M,N}$ with standard methods.

State of the art

	formulation	codes	convergence proof
DDE	✓	✓	✓ [7, 8]
RE	✓	✓	✓ [9]
NRE	✓	✓	✗
NDDE	✗	✗	✗

[7] Breda, Maset, and Vermiglio, *Approximation of eigenvalues of evolution operators for linear retarded functional differential equations*, SIAM J. Numer. Anal., 50 (2012), pp. 1456–1483, DOI:10.1137/100815505.

[8] Breda, Maset, and Vermiglio, *Stability of linear delay differential equations. A numerical approach with MATLAB*, Springer Briefs Control, Autom. and Robot., Springer, New York, 2015, DOI:10.1007/978-1-4939-2107-2.

[9] Breda and Liessi, *Approximation of eigenvalues of evolution operators for linear renewal equations*, SIAM J. Numer. Anal., 56 (2018), pp. 1456–1481, DOI:10.1137/17M1140534.

Convergence of $\sigma(T_{M,N})$ to $\sigma(T)$ and spectral accuracy

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$\rightsquigarrow$ T is not compact in the norm topology

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$\rightsquigarrow$ T is not compact in the norm topology

- Weaker convergence, possibly only on eigenspaces [10].
- Results from the new perturbation theory [5].
- RE and DDE approximating the NRE.

[5] Diekmann and Verduyn Lunel, *Twin semigroups and delay equations*, arXiv:1906.03409 (2019).
[10] Chatelin, *Spectral Approximation of Linear Operators*, Classics Appl. Math. 65, SIAM, Philadelphia, 2011, DOI:10.1137/1.9781611970678.

Experiments

$$\text{NRE} \qquad x(t) = f(t)x(t-1)$$

$$\text{RE}_\epsilon \qquad x(t) = \frac{f(t)}{2\epsilon}\int_{-1-\epsilon}^{-1+\epsilon} x(t+\theta)\,d\theta$$

$$\text{DDE}_\epsilon \quad x'(t) = \frac{f(t)}{2\epsilon}\big(x(t-1+\epsilon) - x(t-1-\epsilon)\big)$$

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Choices for $f(t)$:

$$\text{CONST} \quad f(t) \equiv 2$$

$$\text{SIN} \quad f(t) = \sin(2\pi t)$$

$$\text{STEP} \quad f(t) = \begin{cases} 1, & t \in k + \left]0, \frac{1}{2}\right[\\ 0, & t \in \frac{k}{2} \\ -1, & t \in k + \left]\frac{1}{2}, 1\right[\end{cases}$$

$$\text{EXP} \quad f(t) = \exp(t - \lfloor t \rfloor)$$

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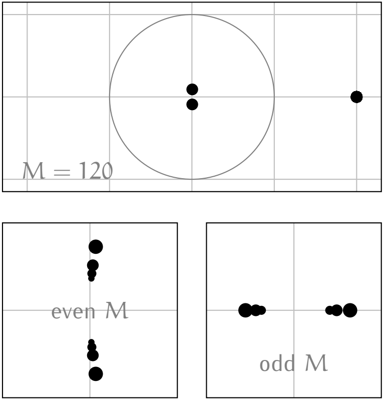
$$T_M \coloneqq T_{M,M}$$

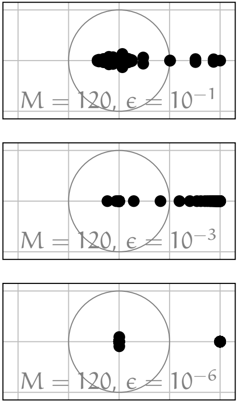
Experiments: CONST

Expectation: $\sigma(T) = \{2\}$

NRE: $\sigma(T_M) \approx \{0, 2\}$



RE$_\epsilon$: $\sigma(T_M) \to \{0, 2\}$

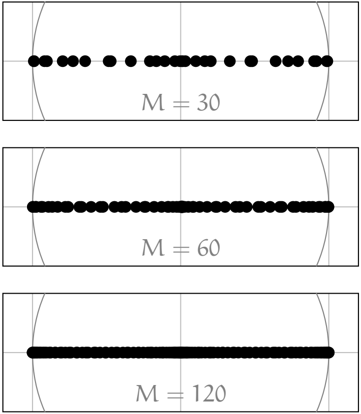


DDE$_\epsilon$: $\sigma(T_M) \to \{0, 1, 2\}$

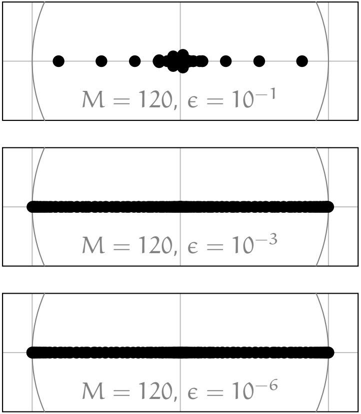
Experiments: SIN

Expectation: $\sigma(T) = ???$

NRE: $\sigma(T_M) \approx [-1, 1]$



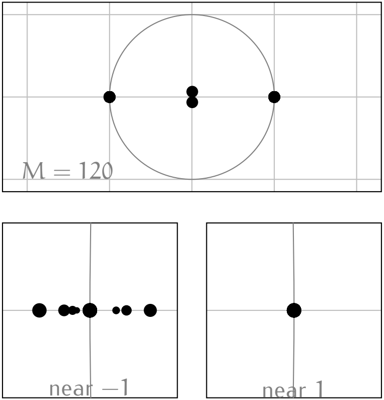
RE_ϵ, DDE_ϵ: $\sigma(T_M) \to [-1, 1]$



Experiments: STEP

Expectation: $\sigma(T) = ???$

NRE: $\sigma(T_M) \approx \{-1, 0, 1\}$



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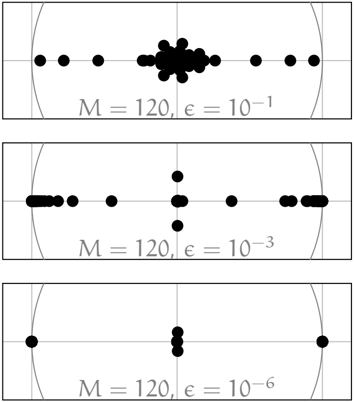
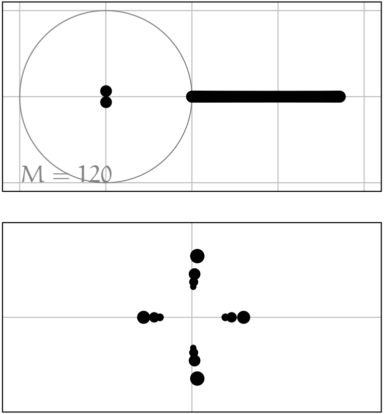


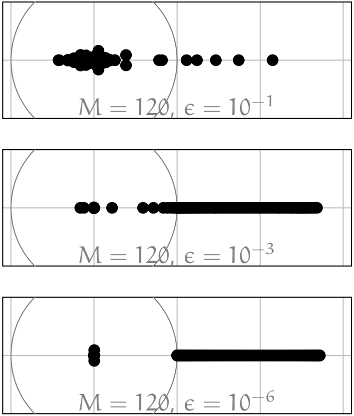
Experiments: EXP

Expectation: $\sigma(T) = ???$

NRE: $\sigma(T_M) \approx \{0\} \cup [1, e]$

RE$_\epsilon$, DDE$_\epsilon$:
$\sigma(T_M) \to \{0\} \cup [1, e]$





Conjecture

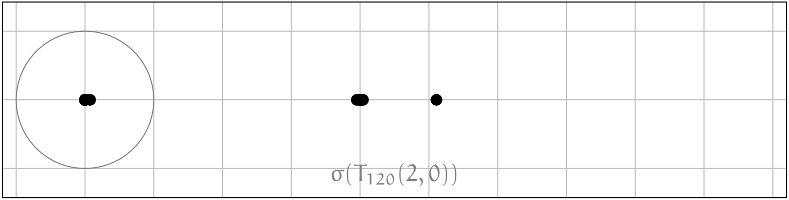
$$\sigma(T) = f([0,1]) = f(\mathbb{R})$$

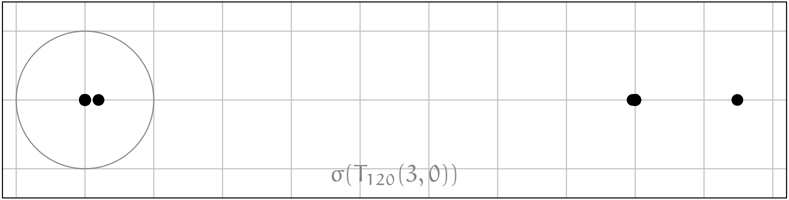
and

$$\sigma(T_M) \to \{0\} \cup f([0,1]) = \{0\} \cup f(\mathbb{R})$$

Expectations: $\sigma(T(2,0)) = \sigma(T^2) = (\sigma(T))^2 = \{4\}$,
$\sigma(T(3,0)) = \{8\}$

NRE, RE_ϵ:



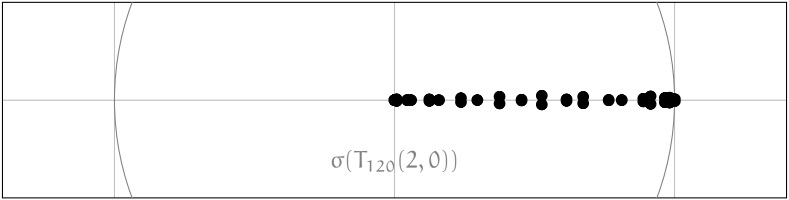


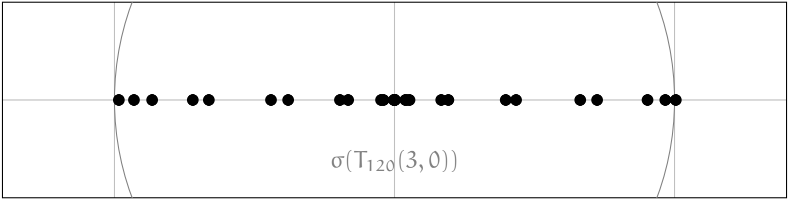
DDE_ϵ: $\cdots \cup \{1\}$

Experiments: SIN, 2–3 periods

Expectations: $\sigma(T_M(2,0)) = [0,1]$, $\sigma(T_M(3,0) = [-1,1]$

NRE, RE_ϵ, DDE_ϵ:





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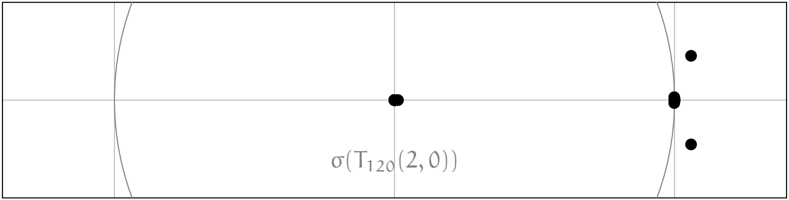

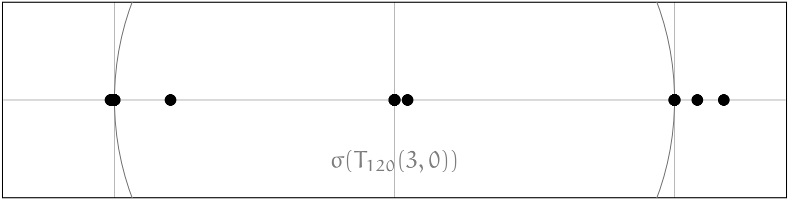

Experiments: EXP, 2–3 periods

Expectations: $\sigma(T_M(2,0)) = \{0\} \cup [1, e^2]$,
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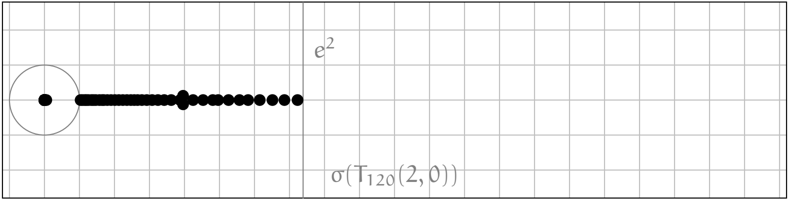
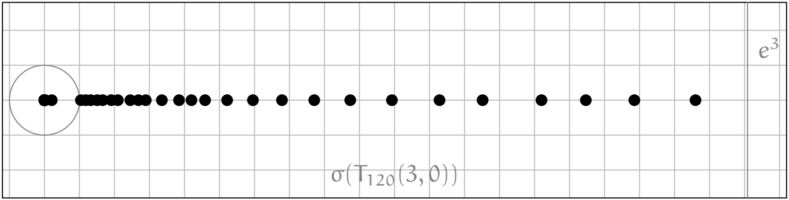
NRE, RE_ϵ, DDE_ϵ:





Open questions

- $\sigma(T) = f(\mathbb{R})$ and $\sigma(T_M) = \{0\} \cup f(\mathbb{R})$?
- Presence and multiplicity of 0.
- Extra eigenvalues of $\sigma(T_M(k, 0))$: spurious or "true"?
- Convergence of $\sigma(T_M)$ to $\sigma(T)$ and convergence rate.
- Convergence of RE_ϵ and DDE_ϵ to NRE.
- New perturbation theory: principle of linearised stability, Floquet theory, regularity of eigenfunctions.
- Are eigenfunctions smooth?

Thanks to

GNCS

INdAM

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Reconstructing the solution of an RE

$$\begin{cases} x(t) = L(t)x_t \\ x_0 = \varphi \end{cases} \qquad T \coloneqq T(p, 0) \colon \varphi = x_0 \mapsto x_p$$

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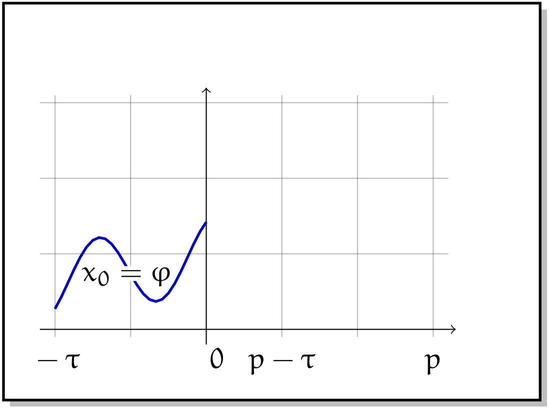

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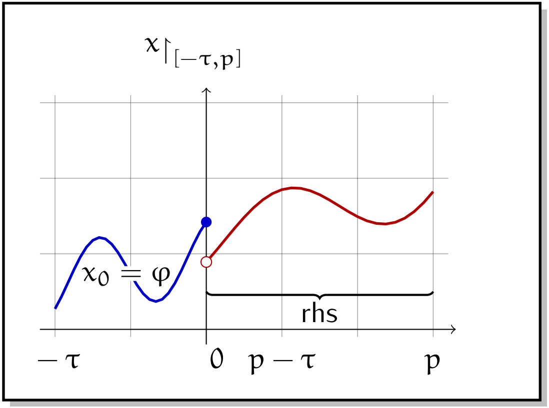

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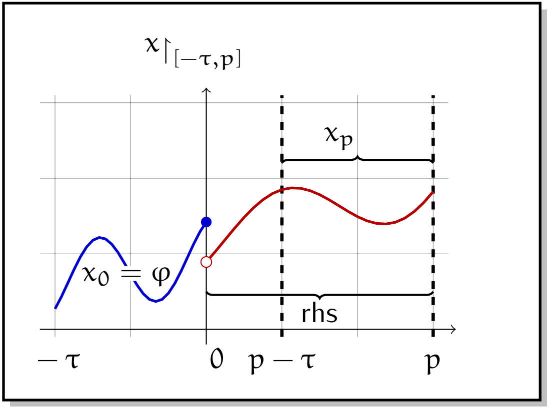

Reconstructing the solution of an RE

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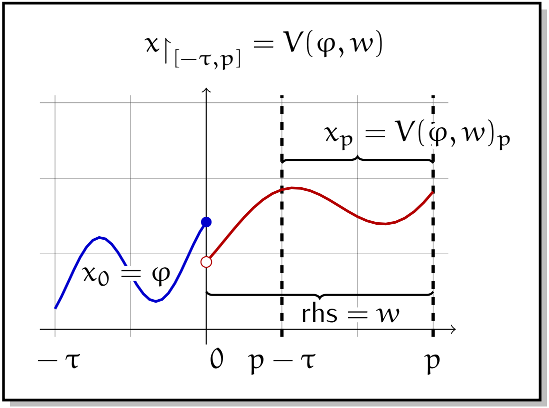

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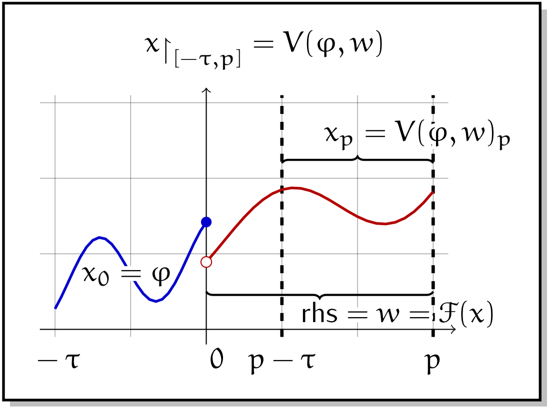

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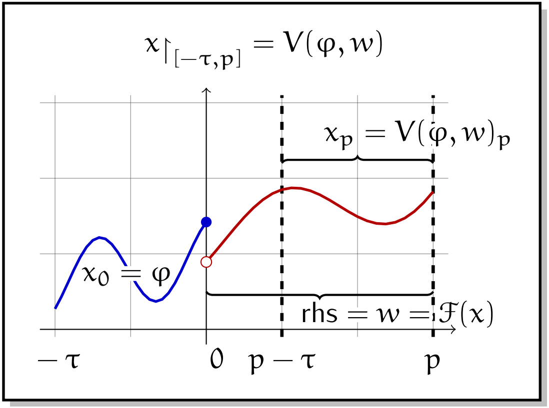

$$T\varphi = V(\varphi, w)_p$$
$$w = \mathcal{F}V(\varphi, w)$$

Reconstructing the solution of a DDE

$$\begin{cases} x'(t) = L(t)x_t \\ x_0 = \psi \end{cases} \qquad T \coloneqq T(p, 0)\colon \varphi = x_0 \mapsto x_p$$

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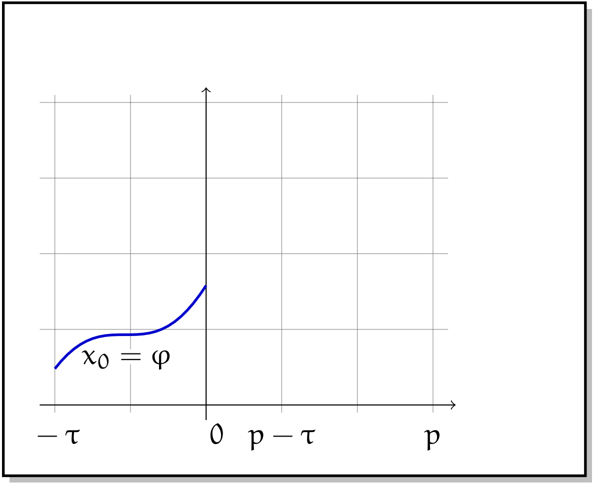

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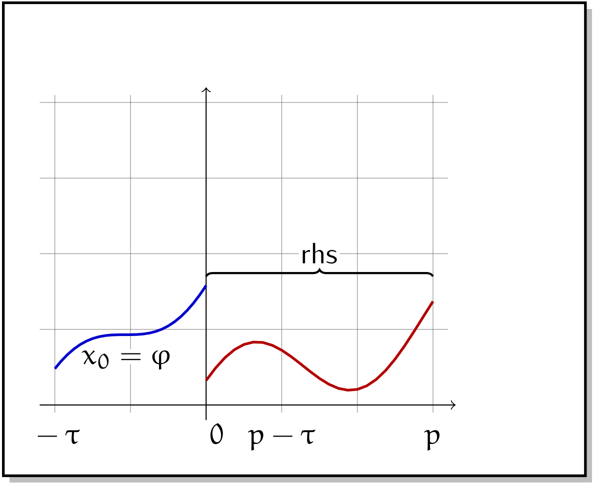

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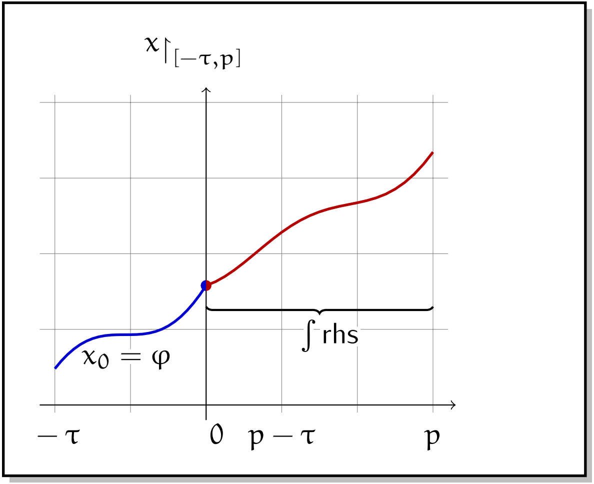

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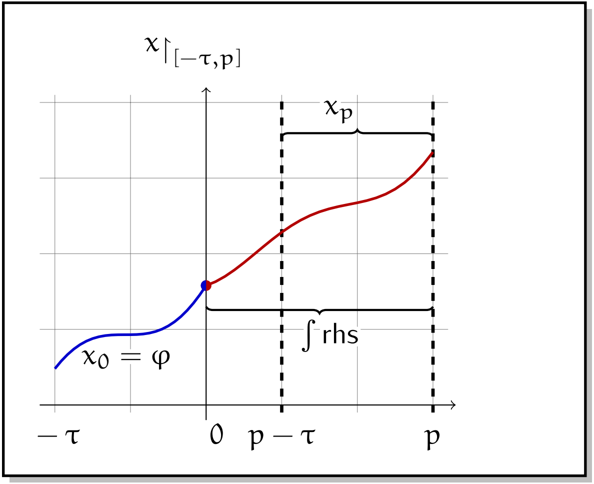

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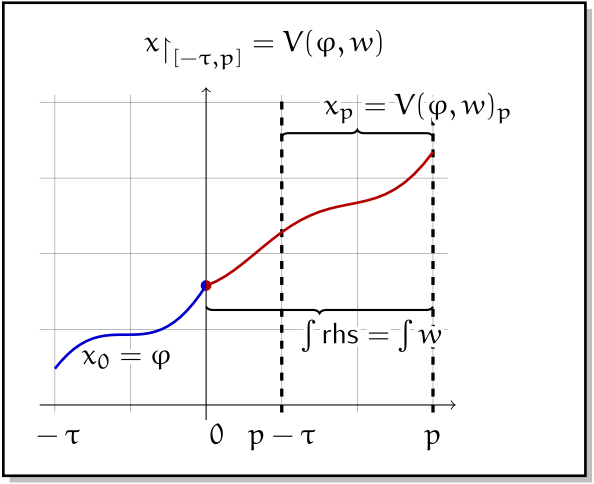

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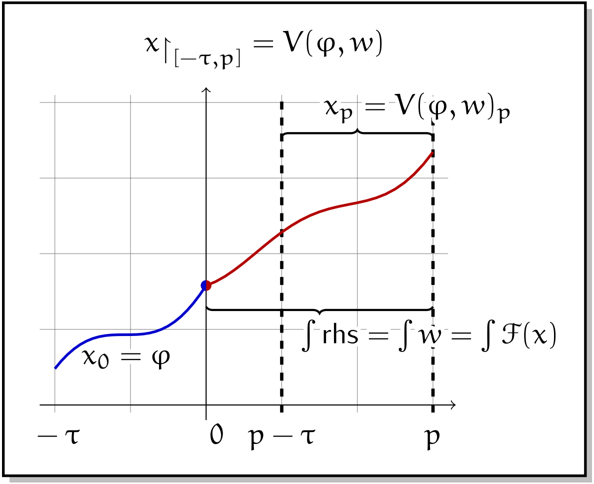

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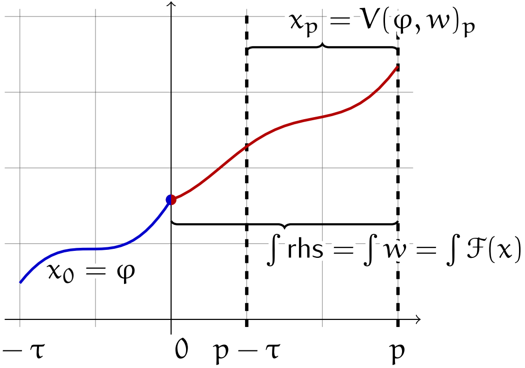
$$x\restriction_{[-\tau,p]} = V(\varphi, w)$$



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