INdAM-COFUND-2012 - call 2 - Final Report

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Outgoing Host Institution: King's College London, London (GB) Return Host Institution: Università degli Studi di Parma, (IT) Period: 1 December 2015 - 30 November 2017 Project Title: Towards Hida Theory over Global Function Fields

Research activity

The project was mainly to establish foundations for the construction of families of Drinfeld modular forms. In particular, we investigated the relation between the analogue of the Atkin-Lehner U_p operator and its eigenvalues.

Let $N, k \in \mathbb{Z}_{\geq 0}$ and denote by $S_k(N)$ the \mathbb{C} -vector space of cuspidal modular forms of level N and weight k. Hecke operators T_n , $n \geq 1$, are defined on $S_k(N)$ and when $p|N, T_p$ is also known as the Atkin, or Atkin-Lehner U_p -operator.

The U_p -operator is well known in literature for detecting those modular forms which belong to a *p*-adic family. Let *f* be an eigenform of U_p and α its eigenvalue. The *slope* of *f* is defined to be the *p*-adic valuation of α , say $v_p(\alpha)$. Coleman in [4] proved the existence of a lot of *p*-adic families thanks to the fact that *overconvergent* modular forms of small slope are classical. Moreover, eigenforms of slope zero are the so-called *ordinary* forms which play a crucial role in Hida theory ([7], [8]).

When p is a prime number not dividing N, using Petersson inner product, the action of T_p is semisimple on cusp forms. This is no longer true for U_p which fails to be diagonalizable.

In the papers [2] and [3] we studied the analogue of U_p in the realm of global function fields, more precisely $\mathbb{F}_q(t)$, for Drinfeld modular forms.

In our context, the lack of an adequate analogous of Petersson inner product leaves open the question about the diagonalizability of the Hecke operators. For the analogue of the T_p some partial answers were given by Li and Meemark in [9] and Böckle and Pink in [1]: in contrast to the characteristic 0 case these operators are not always diagonalizable. We focused on the operator U_t on forms of level t and, in particular, on the slopes of its eigenforms. We provided a detailed study for U_t (along the lines of [6]) exploiting its relations with degeneracy maps (from level 1 to level t) and trace maps (the other way around).

In number fields there is a direct relation between the Hecke eigenvalues and the Fourier coefficients of a given modular form. In the function field setting, even if it is still possible to associate with every Drinfeld modular form a power series expansion with respect to a canonical uniformizer u at the cusp at infinity (see [5]), the action of the Hecke operators on expansions is not well understood and difficult to handle.

In order to avoid this problem we exploited a different reformulation of Drinfeld cusp forms. The Bruhat-Tits tree \mathcal{T} is a combinatorial counterpart of the Drinfeld upper half plane Ω . For any arithmetic subgroup Γ of $GL_2(\mathbb{F}_q[t])$, cusp forms for Γ have a reinterpretation as Γ -invariant harmonic cocycles, i.e., $S_{k,m}^1(\Gamma) \simeq C_{k,m}^{har}(\Gamma)$.

We are mainly interested in the case $\Gamma = \Gamma_0(t)$ (or $\Gamma_0(\mathfrak{m})$ in general), but computations are more feasible for $\Gamma = \Gamma_1(t)$. Since a Hecke action can be carried out on harmonic cocycles as well, we used this combinatorial interpretation to get the matrices corresponding to U_t : their coefficients are binomials depending only on the weight of the space of cusp forms and on q.

An explicit description of the subspace $S_{k,m}^1(\Gamma_0(t))$ inside $S_k^1(\Gamma_1(t))$ (we do not mention the type here because it is not relevant since all matrices in $\Gamma_1(t)$ have trivial determinant) and a careful study of these binomial coefficients allow us to study the diagonalizability of U_t on $\Gamma_0(t)$ -invariant cusp forms in some nontrivial cases or, at least, to perform a computer search on characteristic polynomials and on slopes of eigenforms.

More precisely, the contents of the research paper [2] are the following.

In [2, Section 3] we studied slopes for eigenforms for the action of U_t on $S^1_{k,m}(\Gamma_0(t))$, by dividing cuspidal forms of level t into oldforms (i.e., arising from $S^1_{k,m}(\Gamma_0(1))$) and newforms (which turn out to have all the same slope $m - \frac{k}{2}$). Here we were able to say something on diagonalizability only in the very special case of odd weight k in even characteristic, using of the presence of an inseparable eigenvalue among newforms.

In [2, Section 4] we moved to $\Gamma = \Gamma_1(t)$ and computed the matrix associated with U_t with respect to the basis $\mathcal{B}_k^1(\Gamma_1(t)) \coloneqq \{\mathbf{c}_j(\overline{e}), 0 \leq j \leq k-2\}$ of $S_k^1(\Gamma_1(t))$ (see [2, Section 4.2]), where \mathbf{c}_j are harmonic cocycles and \overline{e} is a particular edge of the fundamental domain. The crucial formula is

$$U_{t}(\mathbf{c}_{j}(\overline{e})) = -(-t)^{j+1} \binom{k-2-j}{j} \mathbf{c}_{j}(\overline{e}) - t^{j+1} \sum_{h \neq 0} \begin{bmatrix} \binom{k-2-j-h(q-1)}{-h(q-1)} \\ +(-1)^{j+1} \binom{k-2-j-h(q-1)}{j} \end{bmatrix} \mathbf{c}_{j+h(q-1)}(\overline{e}).$$
(1)

As it is easy to see from equation (1), the \mathbf{c}_j can be divided into classes modulo q-1 and these classes $C_j \coloneqq \{\mathbf{c}_{\ell} : \ell \equiv j \pmod{q-1}\}$ are stable under the action of U_t . The matrix associated with U_t has (at most) q-1 blocks and U_t is diagonalizable if and only if each block is. We detected the classes C_j associated with $\Gamma_0(t)$ -invariant cuspforms (see [2, Section 4.3]) and then used formula (1) to find coefficients for the associated block M_j . Those matrices turned out to have certain symmetries which are summarized in [2, Section 4.5].

In Section 5 of [2] we studied the M_j associated to $S^1_{k,m}(\Gamma_0(t))$ in some detail. In particular, in the nontrivial cases where the dimension of M_j is at least 2, we showed that

- if $\dim(M_i) \leq j + 1$, then M_i is antidiagonal and it is diagonalizable if and only if q is odd;
- if j = 0 and dim $(M_0) \leq q + 2$, then M_j is diagonalizable unless q is even and dim $(M_0) \geq 4$;
- if $\dim(M_j) \leq 4$ and q is odd, then M_j is diagonalizable.

As dim (M_j) grows it becomes harder to find a pattern for the matrices M_j , so we continued our investigation via some computer search. In Section 6 we provided the links to the files containing the outcome of our computations on characteristic polynomials and slopes for the action of U_t on $S_{k,m}^1(\Gamma_0(t))$ (still seen as a subspace of $S_k^1(\Gamma_1(t))$) and some speculations on what might be worth of more investigation in the future (in particular Conjecture 6.1).

In the paper [3] we addressed the diagonalizability on the whole $\Gamma_1(t)$ -invariant space (at least in small weights). We also studied diagonalizability of U_t on $S_k^1(\Gamma(t))$. We showed that U_t has a large kernel which actually reduces its action to the $\Gamma_1(t)$ case (plus a lot of zeroes, see [3, Theorem 5.1]). This also means that if one wants to study all possible slopes as the congruence subgroup varies, it is sufficient to stop at $\Gamma_1(t)$.

References

[1] G. BÖCKLE 'An Eichler-Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals', available at

www1.iwr.uni-heidelberg.de/groups/arith-geom/home/members/gebhard-boeckle/publications/.

- [2] A. BANDINI, M. VALENTINO On the Atkin U_t operator for $\Gamma_0(t)$ -invariant Drinfeld cusp forms, submitted. arXiv:1710.01038 [math:NT] (2017)
- [3] A. BANDINI, M. VALENTINO On the Atkin U_t operator for $\Gamma_1(t)$ -invariant Drinfeld cusp forms, submitted. arXiv:1710.01036 [math:NT] (2017)
- [4] R. COLEMAN 'Classical and overconvergent modular forms', Invent. Math. 124 (1996), no. 1-3, 215-241.
- [5] E.U. GEKELER 'On the coefficients of Drinfeld modular forms', Invent. math. 93 (1988) 667–700.
- [6] F. GOUVÊA B. MAZUR 'Families of modular eigenforms', Math. Comput. 58 (1992), no. 198, 793-805.
- [7] H. HIDA, 'Galois representations into $GL_2(Z_p[[X]])$ attached to ordinary cusp forms', *Invent.* Math. 85 (1986), no. 3, 545–613.
- [8] H. HIDA, 'Iwasawa modules attached to congruences of cusp forms', Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 2, 231–273.
- W-C. W. LI Y. MEEMARK 'Hecke operators on Drinfeld cusp forms', J. Number Theory 128 (2008), no. 7, 1941–1965.

Conferences, Workshops and Seminars

- Invited talks at conferences/workshops
 - November 14th, 2017 "Atkin U_t -operator for Drinfeld cusp forms: slopes and diagonalizability" at the workshop "Arthmetic over function fields", in Milan (IT).
 - June 26-30, 2017 "Diagonalizability of the U-operator for Drinfeld cusp forms of congruence subgroups" at the conference "Arithmetic of Function Fields", dedicated to the memory of David Goss, in Münster (DE).
- Invited research seminars
 - April 5th, 2017 "On Drinfeld cusp forms and the Atkin-Lehner operator", Xi'an Jiaotong-Liverpool University, Suzhou (CN).
 - November 29th, 2016 "On the diagonalizability of the Atkin U-operator for Drinfeld cusp forms", University of Konstanz (DE), series "Konstanz Women in Mathematics".
 - June 22nd, 2016 "On the diagonalizability of the Atkin U-operator for Drinfeld cusp forms", King's College London, London (GB), series "London Number Theory seminars".
 - February 23rd, 2016 "Euler characteristic for Selmer groups over global function fields" at University of Cambridge (GB).
- Other workshops and conferences

July 24-28, 2017 Congress "Iwasawa 2017" at University of Tokyo, Tokyo (JP).

June 7th, 2017 "INdAM day 2017" at University of Messina (IT).

- 26 June-1 July, 2016 Workshop "New Directions in Iwasawa Theory" at Banff International Research Station, Alberta (CA).
- June 1st, 2016 Workshop "p-adic L-functions day in Cambridge" at Cambridge University (GB).

Invited research stays

April 3-9, 2017 Visiting researcher at Xi'an Jiaotong-Liverpool University, Suzhou (CN); hosted by Prof. I. Longhi.

Hosted visitors

October 16-21, 2016 Prof. Andrea Bandini from Università degli Studi di Parma.

May 21-27, 2017 Prof. Stefano Vigni from Università di Genova.

Publications

- M. VALENTINO On Euler characteristics of Selmer groups for abelian varieties over global function fields, Arch. Math. 106 (2016), 117–128. DOI Number: 10.1007/s00013-015-0858-y.
- 2. A. BANDINI, M. VALENTINO Euler Characteristic and Akashi Series for Selmer groups over global function fields, submitted. arXiv:1705.04703 [math:NT] (2016)
- **3.** A. BANDINI, M. VALENTINO On the Atkin U_t operator for $\Gamma_0(t)$ -invariant Drinfeld cusp forms, submitted. arXiv:1710.01038 [math:NT] (2017)
- **4.** A. BANDINI, M. VALENTINO On the Atkin U_t operator for $\Gamma_1(t)$ -invariant Drinfeld cusp forms, submitted. arXiv:1710.01036 [math:NT] (2017)

Teaching

A.Y. 2016/2017: Tutor in Galois Theory - Department of Mathematics, King's College London.

Other professional activity

- In 2016 I started to be a reviewer for AMS;
- For the A.Y. 2016/2017 I was a member of the interview panel for admission to PhD program of LSGNT (London School of Geometry and Number Theory).

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