COFUND 2012 INdAM Fellowships in Mathematics and/or Applications fields

Report: "p-adic cohomology and homotopy theory for equicharacteristic local fields"

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1 Scientific Outcomes

The primary goal of the proposed project was to fully develop the theory of rigid cohomology for local fields in positive characteristic, and to use this to study questions in the arithmetic of varieties over local and global function fields. This goal was more or less fully achieved, with almost all of the prospective results described in the project proposal now theorems.

The first major achievement was the publication of the monograph [1]. In it, we established all the basic foundational results that are required of a good cohomology theory: finite dimensionality, Poincaré duality, Gysin isomorphisms, excision exact sequences, and a the existence of a well-behaved category of local systems. Key arithmetic results such as a p-adic weight-monodromy conjecture (following Crew) were also proved.

Having successfuly set-up the foundations of p-adic cohomology for varieties over local fields, I then explored a number of applications to the study of degenerating varieties in positive characteristic. For example, in [2] we showed that the Gauss–Manin connection in p-adic cohomology controls the shape of a combinatorial degeneration of surfaces in characteristic p, mirroring existing results in ℓ-adic or Betti cohomology. Similar results for curves, but using unipotent fundamental groups instead of cohomology, were proved in [3].

One of the key properties of this new p-adic cohomology was shown in [7], namely that it contains at least as much arithmetic information as that found in ℓ-adic étale cohomology, in the sense that it computes the same Frobenius eigenvalues. Another application was given in [4], where we provided a new formulation and proof, as well as a generalisation to the semi-stable case, of a result on Morrow concerning liftability of line bundles in terms of deformations of their Chern classes. In fact, this result illustrates one of the reasons why it is so important to have a robust theory of p-adic cohomology, as this deformation information cannot be seen in the ℓ-adic theory.

There are some goals of this project that have not been completed: for example the study of the period map coming from the unipotent fundamental group. A concrete description of this map in terms of a suitable version of 'Coleman functions', as well any prospective applications of this to the study of rational points via Kim's method, are still to be realised. Nor did I pursue the goal of expanding the categories of coefficients introduced in [1] to arithmetic

$\mathcal{D}^\dagger$-modules capable of supporting Grothendieck's 6 operations: this has in fact formed part of work-in-progress of R. Crew, and joint work-in-progress of D. Caro and D. Vauclair.

On top of the original aims as described in the proposal, there were other successful research outcomes achieved over the course of the fellowship. In [5] we adapted an approach of dos Santos in order to prove exactness of the homotopy sequence for overconvergent p-adic fundamental groups. This was then used to deduce a weak version of the Lefschetz hyperplane theorem, as well as a comparison between convergent and étale fundamental groups. In another direction, in [6] we studied the problem of degenerations of K3 surfaces more closely, giving a criterion for good reduction analogous to the Néron–Ogg–Shafarevich criterion for abelian varieties.

Looking towards the future, I hope to return to some of the unresolved goals of this project, in particular to developping a potential analogue of the theory of Coleman integration in equicharacteristic. There are also many interesting questions arising from research carried out during this fellowship that I hope to be able to explore further. For example, the results of [4] hint that the cohomology theory developed in [1] could provide the necessary tools in order to prove an equicharacteristic version of a theorem of Maulik and Poonen on the Picard number jumping locus. It would also be interesting to extend the homotopy exact sequence from [5] to higher homotopy groups, the recent striking results of Achinger on $K(\pi, 1)$-spaces in positive characteristic suggesting that in fact this sequence should be exact on the left in many cases.

2 Publications and preprints

Below is a list of all books, journal articles and preprints published as sole or joint author during the course of the fellowship.

[1] Christopher Lazda and Ambrus Pál, *Rigid Cohomology over Laurent Series Fields*, Algebra and Applications, vol. 21, Springer, 2016, `http://dx.doi.org/10.1007/978-3-319-30951-4`.

[2] Bruno Chiarellotto and Christopher Lazda, *Combinatorial degenerations of surfaces and Calabi–Yau threefolds*, Algebra Number Theory **10** (2016), no. 10, 2235–2266, `http://dx.doi.org/10.2140/ant.2016.10.2235`.

[3] Christopher Lazda, *Fundamental groups and good reduction criteria for curves over positive characteristic local fields*, preprint (2016), `https://arXiv.org/abs/1604.06024`, to be published by J. Théor. Nombres Bordeaux.

[4] Christopher Lazda and Ambrus Pál, *Cycle classes in overconvergent rigid cohomology and a semistable Lefschetz* $(1,1)$ *theorem*, preprint (2017), `https://arxiv.org/abs/1701.05017`.

[5] Christopher Lazda and Ambrus Pál, *A homotopy exact sequence for overconvergent isocrystals*, preprint (2017), `https://arxiv.org/abs/1704.07574`.

[6] Bruno Chiarellotto, Christopher Lazda, and Christian Liedtke, *A Néron–Ogg–Shafarevich criterion for K3 surfaces*, preprint (2017), `https://arXiv.org/abs/1701.02945/`.

[7] Bruno Chiarellotto and Christopher Lazda, *Around ℓ-independence*, Compos. Math. **154** (2018), no. 1, 223–248, https://doi.org/10.1112/S0010437X17007527.

3 Research activity and meeting attendance

During my time in Padova I attended a number of meetings, both to present my own work and to keep up-to-date with as many new developments in the field as possible. Invitations to speak at local seminars in Milan, Paris, Poznań and Strasbourg allowed me to disseminate the results I obtained during this fellowship to a wide, international audience. At the conference on "p-adic Analytic Geometry and Differential Equations", held at the CIRM research centre in Luminy in 2017, I was able to present some of my results to a broad spectrum of some of the leading researchers working in the subject.

As well as collaborating successfully with Prof. Bruno Chiarellotto, my local mentor at the Universitdi Padova, on several research articles [2, 6, 7], I continued to work with my Ph.D. advisor Dr. Ambrus Pál (Imperial College London), resulting in the preprints [4, 5]. I developed new collaborations with Prof. Christian Liedtke (TU München), leading to or work on degenerations of K3 surfaces [6], as well as with Dr. Nicolà Mazzari (Bordeaux), in ongoing work on p-adic realisations of motives. Individual research visits to Rennes and Grenoble, as well as a week at the Mittag-Leffler institute in Stockholm as part of the research program on "Algebro-Geometric and Homotopical Methods" all presented an opportunity to establish new mathematical relationships and potential new collaborations, for example with Prof. Bernard Le Stum (Rennes I) on his new approach to a general form of overconvergent cohomology, and with Prof. Andrea Pulita (Grenoble) on generalising the p-adic local monodromy theorem.

I also played a role in organising a successful workshop entitled "p-adic cohomology and arithmetic applications", held at the Banff International Research Centre in Canada, and organised jointly with Dr. Tomoyuki Abe (Kavli IPMU), Dr. Pál and Prof. Kiran Kedlaya (UC San Diego). This was held in October 2017, shortly after I left my position at the Università di Padova.

4 Visiting period

I began my fellowship in November 2015, and stayed in Padova for a total of 22 months, finishing at the end of August 2017. I left in order to take up a new post-doctoral position at the University of Amsterdam.

5 Other activities

I helped to run a research seminar in Spring 2016 at the Univeristá di Padova on the moduli of p-divisible groups, following a recent article of Scholze and Weinstein. I also occasionally assisted Prof. Chiarellotto with graduate teaching as part of the ALGANT program in Padova.