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Final Project Report GRIP

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Academic Results

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\overline{\mathbb{F}_p}$ of the finite field of prime cardinality $p > 0$ and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-rational structure $\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\} \leqslant \mathbf{G}$; note q is a power of p. The group $\mathbf{G}^F$ is a finite group known as a *finite reductive group*. The principal example of a finite reductive group is $\mathrm{GL}_n(q)$ obtained from the algebraic group $\mathrm{GL}_n(\overline{\mathbb{F}_p})$ as the fixed points under the endomorphsim $(x_{ij}) \mapsto (x_{ij}^q)$.

An element of $\mathbf{G}$ is called *unipotent* if its order is a power of p; for example a unipotent matrix in $\mathrm{GL}_n(\overline{\mathbb{F}_p})$. Assuming that p is a *good prime* for $\mathbf{G}$ Kawanaka has associated to every unipotent element $u \in \mathbf{G}^F$ a complex character $\Gamma_u : \mathbf{G}^F \to \mathbb{C}$ of the finite group $\mathbf{G}^F$ called a generalised Gelfand–Graev character (GGGC). If $\mathrm{Irr}(\mathbf{G}^F)$ is the set of complex irreducible characters of $\mathbf{G}^F$ then for any $u \in \mathbf{G}^F$ we have

$$\Gamma_u = \sum_{\chi \in \mathrm{Irr}(\mathbf{G}^F)} \langle \Gamma_u, \chi \rangle \chi,$$

where $\langle \Gamma_u, \chi \rangle \geqslant 0$ is a non-negative integer. One of the main concerns of project GRIP is to obtain a better understanding of the integers $\langle \Gamma_u, \chi \rangle$ assuming $Z(\mathbf{G})$ is connected. The paper [Tay16b] makes a significant advance towards this problem.

GGGCs in small characteristics

To describe the main results of [Tay16b] we introduce some notation. For any element $u \in \mathbf{G}$ we denote by $\mathcal{O}_u$ the $\mathbf{G}$-conjugacy class containing u and by $A_{\mathbf{G}}(u)$ the finite group $C_{\mathbf{G}}(u)/C_{\mathbf{G}}^\circ(u)$ where $C_{\mathbf{G}}(u)$ is the centraliser of u in $\mathbf{G}$ and $C_{\mathbf{G}}^\circ(u)$ is its connected component. If $\mathcal{O}$ and $\mathcal{O}'$ are two unipotent $\mathbf{G}$-conjugacy classes then we set $\mathcal{O} \leqslant \mathcal{O}'$ if $\mathcal{O} \subseteq \overline{\mathcal{O}'}$ where $\overline{\mathcal{O}'}$ is the Zariski closure of $\mathcal{O}'$. The relation $\leqslant$ is then a partial ordering on the set of all unipotent conjugacy classes.

Finally we recall that to every irreducible character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ Lusztig has defined a unique positive integer $n_\chi > 0$ which has the property that $n_\chi \cdot \chi(1)$ is a polynomial in q with integral coefficients. If $p, q \gg 0$ are sufficiently large then the following result is due to the combined efforts of Achar–Aubert [AA07], Geck–Malle [GM00], Kawanaka [Kaw85; Kaw86], and Lusztig [Lus92]. The main result of [Tay16b] is that the statement is still valid assuming only that p is a good prime for $\mathbf{G}$.

Theorem (Achar–Aubert, Geck–Malle, Kawanaka, Lusztig, T.). *Assume p is a good prime for $\mathbf{G}$ then for any irreducible character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ there exists a unique F-stable unipotent conjugacy class $\mathcal{O}_\chi^* \subseteq \mathbf{G}$ such that the following hold:*

(a) $\langle \Gamma_u, \chi \rangle \neq 0$ *for some* $u \in \mathcal{O}_\chi^{*F}$,

(b) if $\langle \Gamma_u, \chi \rangle \neq 0$ then $\mathcal{O}_u \leqslant \mathcal{O}_\chi^*$.

Moreover, if $Z(\mathbf{G})$ is connected then for any $u \in \mathcal{O}_\chi^{*F}$ we have n_χ divides $|A_{\mathbf{G}}(u)|$ and

$$0 \leqslant \langle \Gamma_u, \chi \rangle \leqslant \frac{|A_{\mathbf{G}}(u)|}{n_\chi}.$$

The class $\mathcal{O}_\chi^\star$ is called the *wave front set* of χ and was conjectured to exist by Kawanaka in [Kaw85]. Thus the above result gives a complete proof of Kawanaka's conjecture. This result has already been applied by Dudas–Malle [DM16] as part of their program for determining new decomposition matrices for groups of exceptional type.

Action of automorphisms and the McKay conjecture

One of the motivations for this project was the recent activity on the *McKay conjecture*. To state this conjecture let us introduce some notation. If G is a finite group and ℓ is a prime then we denote by $\mathrm{Irr}_{\ell'}(G)$ the set of irreducible characters $\chi \in \mathrm{Irr}(G)$ whose degree $\chi(1)$ is coprime to ℓ. The McKay conjecture then states that for any finite group G and prime ℓ we have $|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))|$ where $P \leqslant G$ is a Sylow ℓ-subgroup of G. This conjecture, while simple to state is vexingly difficult to prove.

In their landmark paper [IMN07] Isaacs–Malle–Navarro proposed a new approach to this conjecture. Their main result states that if a family of conditions, known as the inductive McKay condition, hold for each quasisimple group G then the McKay conjecture holds for all finite groups. This is a significant reduction of the problem as much is known about finite quasisimple groups in comparison to an arbitrary finite group. For instance, Malle–Späth [MS16] have successfully applied this method to prove the McKay conjecture holds for the prime $\ell = 2$. However, the case where ℓ is odd is still open.

If $\chi \in \mathrm{Irr}(G)$ is an irreducible character of a finite group and $\sigma : G \to G$ is an automorphism then ${}^\sigma\chi \in \mathrm{Irr}(G)$ is again an irreducible character, where ${}^\sigma\chi(g) = \chi(\sigma^{-1}(g))$. This gives an action of the automorphism group $\mathrm{Aut}(G)$ of G on $\mathrm{Irr}(G)$. As part of the inductive McKay condition one needs to understand this action. In particular, one wants to know the stabiliser of an irreducible character under this action. In [Tay16a] I have completely solved this problem when G is a finite symplectic group defined over a field of odd characteristic, using my results on GGGCs. Specifically I obtained the following.

Theorem (T.). *Let G, resp., $G^\star$, be a finite symplectic group $\mathrm{Sp}_{2n}(q)$, resp., finite special orthogonal group $\mathrm{SO}_{2n+1}(q)$, with q an odd prime power. If $\sigma : G \to G$ is a field automorphism and $\mathcal{E}(G^\star, s) \subseteq \mathrm{Irr}(G)$ is a Lusztig series labelled by a quasi-isolated semisimple element $s \in G^\star$ then we have ${}^\sigma\chi = \chi$ for all $\chi \in \mathcal{E}(G^\star, s)$.*

From this result one can describe the effect of a field automorphism on any irreducible character using the equivariance of Deligne–Lusztig induction. In practice this can be difficult to compute with but in [Tay16a] I give a completely explicit method for computing this action. This result works for any finite reductive group and should hopefully apply to other series of groups. With these results I was able to deduce that the global portion of the inductive McKay condition holds for the finite symplectic groups.

Decomposing Deligne–Lusztig induction

Now assume $Z(\mathbf{G})$ is connected and $\mathbf{L} \leqslant \mathbf{P} \leqslant \mathbf{G}$ is an F-stable Levi complement of a parabolic subgroup $\mathbf{P} \leqslant \mathbf{G}$. To the pair $(\mathbf{L}, \mathbf{P})$ Deligne and Lusztig have defined a map $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} : \mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$ called Deligne–Lusztig induction. This map sends irreducible characters of $\mathbf{L}^F$ to virtual characters of $\mathbf{G}^F$, i.e., to $\mathbb{Z}$-linear combinations of $\mathrm{Irr}(\mathbf{G}^F)$. The first step of project GRIP was to solve the following problem.

Problem. *Give an explicit combinatorial description of the map* $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$.

This is by far the most complicated step in project GRIP and has taken much longer than expected. This is largely due to the fact that many statements I expected to be known are not proved in the generality that I need. Thus many intermediate results have been required to achieve this goal. For example, in joint work with M. Geck we have given new proofs of unpublished results of Asai [Asa] which are crucial to reducing the above problem to the case where $\mathbf{G}$ has a simple derived subgroup.

Another problem which we have needed to solve concerns Harish-Chandra induction. When the parabolic subgroup $\mathbf{P}$ is F-stable then the map $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$ is known as Harish-Chandra induction and has the property that $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\chi)$ is a character for any $\chi \in \mathrm{Irr}(\mathbf{G}^F)$, i.e., $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\chi)$ is a linear combination of $\mathrm{Irr}(\mathbf{G}^F)$ with non-negative coefficients. In this case the decomposition of $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\chi)$ can be described through the theory of Hecke algebras. In fact, in this way one obtains a labelling of the set $\mathrm{Irr}(\mathbf{G}^F)$ known as the Harish-Chandra labelling. Unfortunately, the labelling of interest to us is the one given by Lusztig in [Lus84] and it is not currently known how to translate one labelling to the other. In recent work I have described the transition between these labellings, which is an important first step in solving the above problem.

Due to the complex nature of solving the above problem I have chosen not to release these intermediate results separately. Instead I will keep everything together and release this as a single document. This will avoid repetition and help maintain a cohesive narrative for the final work. I greatly look forward to the completion of this project.

Associated to any connected reductive algebraic group $\mathbf{G}$ is a finite group known as the Weyl group of $\mathbf{G}$. A naturally occurring problem when addressing the decomposition of $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\chi)$ is to understand the decomposition of induced representations in Weyl groups. Moreover, one must often consider induction on cosets of Weyl groups. A specific problem that has arisen in the context of determining the effect of $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$ is to obtain an analogue of the main result of [Tay15] for coset induction in the case of a twisted Weyl group of type ${}^2\mathsf{D}_n$. This involves a careful determination of certain signs that occur in the definition of the irreducible characters of the coset. A priori it is not clear how to determine these signs combinatorially; so to obtain a better understanding of what these signs are I have written a compute program in C and Python to compute examples of such decompositions. One of the advantages of using C and Python is that one can use arbitrary amounts of system resources, so one can compute very large examples. As part of this work I discovered an improvement to the algorithm presented in [Pfe94] for computing the character tables of type B_n and D_n Weyl groups. This work will appear in the future and will be released to the community through the open source platform GitHub.

Professional Development and Training Objectives

In the first year of the fellowship I took the opportunity to develop my teaching skills. As I have worked in several countries whose native language is not my own it has been difficult to develop a proper teaching portfolio. To improve this I taught a 20 hour graduate course on algebraic groups, together with my colleague Dr Iulian Simion. This was attended by several graduate students from the Department of Mathematics and was well received. During the course I prepared typed weekly lecture notes and maintained a course website. Furthermore at the end of the course Dr Simion and myself wrote a final exam to assess the students. Undertaking such a course was an important training objective for myself and has helped me to develop into a more well rounded researcher.

In addition to teaching a graduate course I was also involved with organising a summer school on characters of finite reductive groups in Les Diablerets, Switzerland. As well as dealing with the practical aspects of organising the meetings I also ran the exercise sessions and wrote several pages of exercises to accompany the lecture courses. This provided me with an additional opportunity to gain experience communicating with graduate students in a teaching capacity.

The academic work described above has helped to achieve two important training objectives. The first was to obtain a deeper understanding of Shintani descent and to understand its role in determining the decomposition of the Deligne–Lusztig induction map $R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$. Although this has taken time it has been an invaluable experience to learn this new topic and to add the technique of Shintani descent to my skill set. The second key training objective was to gain a better understanding of how computational techniques can be used to compute examples in the representation theory of finite reductive groups. Through the work mentioned above I have learned new programming skills and have been able to apply them to computational problems. Thus again adding new important skills to my repertoire.

Just before starting the fellowship in February 2012 I had the opportunity to participate in a 2 week special session in Pisa on Lie Theory and Representation Theory. This provided me an ideal opportunity to meet with other researchers both in Italy and internationally working in related areas. In April 2015 I was also invited to participate in the workshop Representations of Finite Groups held in Oberwolfach (Germany). There I was invited to give a talk about my work and made important connections with Prof. Pham Huu Tiep (USA). My recent work has direct applications to problems that he and his collaborators have been investigating and this provided an excellent opportunity to discuss these issues. I have also participated in several other major conferences to develop my international connections further.

An important goal for any research academic is to develop collaborative projects. Thanks to the financial support of the fellowship I was able to invite two collaborators Olivier Dudas and Olivier Brunat from Paris VII (France) to work on a new research problem which is intimately related to the project. This work remains ongoing. I have also collaborated with Mandi Schaeffer-Fry from MSU Denver (USA) applying GGGCs to a question motivated by the modular representation theory of finite groups. This work is soon to appear as a preprint. These are some of my first collaborative projects and are an important step towards reaching a level of professional academic maturity.

Finally to promote the subject of representation theory of finite reductive groups in Italy I used my grant money to organise a 1-day meeting in Padova. The speakers at the meeting were Meinolf Geck (Stuttgart), Radha Kessar (City), Gunter Malle (TU Kaiserslautern), and Gabriel Navarro (València), who are some of the leading world experts in the representation theory of finite groups. This meeting provided people in Italy with an opportunity to learn more about the recent developments in the representation theory of finite groups and helped to promote intra-disciplinary research. This seemed to be a success.

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