Report on the INdAM-COFUND project "Studies on Free Resolutions and Koszul rings"

Hop D. Nguyen

January 24, 2018

The author of this report, Dang Hop Nguyen, has finished his two year work as a Marie Curie fellow of the INdAM. He has been working at the Univeristy of Genoa from 1st October 2015 to 30th September 2017. Below is a summary of Nguyen's activities during the whole period of the fellowship. Beside the final conclusion, the author will divide his activities in four categories:

(1) Scientific progresses related to the INdAM proposal.

(2) Contribution to Italy-Vietnam scientific relationship.

(3) Invited presentations at seminars and conferences.

(4) Scientific papers completed.

1 Scientific progresses related to the INdAM proposal

1.1 Resolutions over quadric hypersurfaces

In the join work with Conca and Vu [1], we prove the following result.

Theorem 1.1 (Conca, Nguyen, Vu [1, Theorem 5.1])**.** *Let R be a quadric hypersurface over a field k. Then any product of ideals generated by linear forms of R has a linear resolution over R.*

This settles Conjecture 1.2 in Part B of my proposal. It also represents a progress toward the conjecture by Conca, De Negri and Rossi about products of ideals generated by linear forms in an universally Koszul algebra.

1.2 Absolutely Koszul algebras

Let k be a field. For integers $c, n \geq 1$, let $V_{n,c}$ denote the c-th Veronese subring of $k[x_1, \ldots, x_n]$. Recall that Question 1.16 in my proposal asks whether $V_{n,c}$ is absolutely Koszul if and only if one the following cases happens:

(i) $c = 1$, n arbitrary,

(ii) $c = 2, n \leq 6$,

(iii) $c\{3, 4\}, n \leq 4$,

(iv) $c \geq 5, n \leq 3$.

In the paper [3], which will appear in the Journal of Pure and Applied Algebra, I prove the following result:

Theorem 1.2 (Nguyen, [3, Theorem 1.4]). *Let k be a field of characteristic 0. Then $V_{n,c}$ is not absolutely Koszul in the following cases:*

(i) $c = 2, n \geq 7$,

(ii) $c \in \{3, 4\}, n \geq 5$,

(iii) $c = 5, n \geq 4$.

In other words, I have confirmed Question 1.16 for the cases $c \leq 5$. It is not clear whether my techniques can be extended to settle the case $c \geq 6$, namely showing that $V_{n,c}$ is not absolutely Koszul for $n \geq 4$ and $c \geq 6$. Note that if we can show that $V_{4,c}$ is not absolutely Koszul for $c \geq 6$, then since $V_{4,c}$ is an algebra retract of $V_{n,c}$ for any $n \geq 4$, it follows that $V_{n,c}$ is not absolutely Koszul for all such n.

1.3 Powers of ideals

In [2], together with Hà, N.V. Trung and T.N. Trung, I proved the following result:

Theorem 1.3 (Hà, Nguyen, Trung, Trung [2, Theorem 6.7]). *Let $f(n)$ be a convergent non-negative numerical function. Then there exist a polynomial ring R over any given field k, and a monomial ideal I of R such that* $\operatorname{depth} R/I^n = f(n)$ *for all $n \geq 1$.*

This settles a conjecture of Herzog and Hibi from 2005.

In a recent joint work with N.V. Trung [4], I proved a weak analogue of Theorem 1.3 for symbolic powers. More precisely, we show that for any *eventually periodic positive* numerical function $f(n)$, there exist a polynomial ring I and a homogeneous ideal I of R such that $\operatorname{depth} R/I^{(n)} = f(n)$ for all n.

My joint work with Vu [5] provide a new formula for the depth and regularity of powers of sums of ideals. Specifically, for R, S polynomial rings over k, $I \subset R, J \subset S$ homogeneous ideals, and $T = R \otimes_k S$, we show that the formulas

(i) $\operatorname{depth} T/(I + J)^s =$

$$\min_{i \in [1,s-1], j \in [1,s]} \left\{ \operatorname{depth} R/I^{s-i} + \operatorname{depth} S/J^i + 1, \operatorname{depth} R/I^{s-j+1} + \operatorname{depth} S/J^j \right\},$$

(ii) $\operatorname{reg} T/(I+J)^s =$

$$\max_{i\in[1,s-1],j\in[1,s]} \left\{\operatorname{reg} R/I^{s-i} + \operatorname{reg} S/J^i + 1, \operatorname{reg} R/I^{s-j+1} + \operatorname{reg} S/J^j\right\}.$$

hold for all $s \geq 1$, if $\operatorname{char} k = 0$, or if I and J are monomial ideals. This result has been applied by other researchers in recent studies on the regularity of powers of edge ideals of graphs.

2 Contribution to Italy-Vietnam scientific relationship

I have given two research talks as well as a series of four lectures for graduate students at the Departmet of Mathematics in Genoa. Together with Aldo Conca from Genoa, I and T. Vu have written the joint paper [1] which answers partially a question due to Conca, De Negri and Rossi on resolutions over universally Koszul algebras.

I also have had some discussions with Maria Evelina Rossi about the linerity defect, and Matteo Varbaro about depth functions of powers of ideals.

As a postdoc in Genova, I have visited the Vietnam Institute for Advanced Study in Mathematics (VIASM) in Hanoi, Vietnam for three weeks in April and May 2016. During this visit, I gave three seminar research talks at the VIASM and the Hanoi Institute of Mathematics (HIM). My cooperation with Vietnam-based mathematician has resulted in two finished papers [2], [4], written (among others) with Trân N. Trung and Ngô V. Trung from the HIM.

In September 2017, I also gave a lecture at the workshop in Hanoi and Halong, honoring Le Tuan Hoa's 60th birthday. That my visits to Vietnam were productive is witnessed by the production of the papers [2], [4] and [5]. I am positive of having further joint work with Vietnamese colleagues in the future.

3 Invited presentations at seminars and conferences

- 2017

  Sep. 11–15. *Vanishing of maps of Tor and powers of ideals.*
  International Conference on Commutative Algebra and its interaction to Combinatorics, Discrete Geometry and Singularity Theory. Honoring Le Tuan Hoa's 60th birthday. Hanoi & Ha Long (Vietnam).

  Sep. 4–9. *Products of ideals of linear forms in quadric hypersurfaces.*
  School and workshop on syzygies, Trento (Italy).

- 2016

  Aug. 4. *Splitting powers of sums.*
  Oberseminar Algebra, Universität Osnabrück.

Jul. 7–12. *Powers of mixed sums and their linearity defect* (Poster presentation).
Conference "Commutative Algebra and its Interactions with Algebraic Geometry: Tight Closure, Linkage, and Syzygies". Honoring the 65th birthday of Craig Huneke, Ann Arbor, Michigan.

Apr. 28. *Powers of ideals: the case of mixed sums.*
Commutative Algebra Seminar, Vietnam Institute for Advanced Study in Mathematics, Hanoi.

Apr. 20. *Linearity defect of edge ideals.*
Algebra and Number Theory Seminar, Hanoi Institute of Mathematics, Hanoi.

Apr. 19. *Linearity defect and Betti splittings.*
Commutative Algebra Seminar, Vietnam Institute for Advanced Study in Mathematics, Hanoi.

Mar. 2. *Componentwise linearity of some intersection of ideals.*
Algebra and Geometry Seminar, University of Genoa.

- 2015

  Oct. 07–10. *Linearity defect of edge ideals and Fröberg's theorem* (Poster presentation).
  Workshop "Combinatorial and Experimental Methods in Commutative Algebra and Related Fields", Universität Osnabrück.

Beside the above events, I also have participated in other scientific workshops and conferences:

- 2017 Jun. 7. Workshop "INdAM Day", University of Messina.
- 2017 May. 14–19. Workshop "Ordinary and symbolic powers of ideals", Banff International Research Station of Casa Matemática Oaxaca, Mexico (participation by invitation only).
- 2015 Oct. 22–24. Incontro di Algebra Commutativa, Genova.

Durig winter semester 2016/2017, I have delievered four 2-hour lectures in an advanced course for graduate students of University of Genoa. The topic is "Free resolutions and products of ideals".

4 Scientific papers completed

The publications finished during my fellowship are listed below.

References

[1] Aldo Conca, Hop D. Nguyen, and Thanh Vu, *Products of ideals of linear forms in quadric hypersurfaces*, submitted to Proc. Amer. Math. Soc. (2017), arxiv.org/abs/1706.08066.

[2] H. Tài Hà, Hop D. Nguyen, Ngô V. Trung and Trân N. Trung, *Symbolic powers of sums of ideals*, submitted to Amer. J. Math. (2017), arxiv.org/abs/1702.01766.

[3] Hop D. Nguyen, *The absolutely Koszul property of Veronese subrings and Segre products*, to appear in J. Pure Appl. Algebra, doi.org/10.1016/j.jpaa.2017.09.017.

[4] Hop D. Nguyen and Ngô V. Trung, *Depth functions of symbolic powers of homogeneous ideals*, Preprint (2017).

[5] Hop D. Nguyen and Thanh Vu, *Powers of sums and their homological invariants*, submitted to J. Pure Appl. Algebra (2017), arxiv.org/abs/1607.07380.

5 Final conclusion

Overall, I am pleased to say that the INdAM-COFUND fellowship has enabled me to pursue meaningful research, and the many activities enabled by the fellowship have greatly helped expanding my horizon and career opportunity. I am taking advantage of the positive experience during my postdoc in Genoa for the pursuance of a scientific career. After finishing the INdAM fellowship, I am currently holding a postdoctoral position at the Otto von Guericker Universität Magdeburg.